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On the structure of the theory of polar elasticity

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This work presents a study of the formal structure of the theory of finite-strain polar elasticity and thermoelasticity with special attention to the construction of *canonical* balance laws that concern the whole system under study and not only each separate degree of freedom. These are the balance of energy and so-called pseudo-momentum whose (i) local form plays an essential role in the theory of smooth material inhomogeneities (so-called gradient materials), (ii) global form—integrated over a material region—finds a direct application in the corresponding theory of fracture (expression of the energy-release rate and path-independent integrals) and (iii) associated jump equations provide a fruitful application in the theory of the progress of coherent phase-transition fronts obeying the second law of thermodynamics by relating the localized surface entropy and hot heat source to the kinetics of the front and helping one devise a criterion of progress. All corresponding expressions are obtained either in quasi-statics or in full dynamics. The untouched problems are those of completeness of the set of conservations laws, the much expected 'geometrization' of the theory, and the generalization to incoherent phase-transition fronts for which dislocations *and* disclinations will play a foremost role. Hints for the solution of these three problems are given.

Keywords: polar elasticity; inhomogeneities; microstructure; fracture; phase transition; conservation laws

1. Introduction

Recent works dealing with the general problem of fracture and the propagation of phase-transition fronts in deformable materials (in a synthetic form, see Maugin (1993, 1995)) have shown that: (i) the cases of (fully non-dissipative) elasticity and of the thermoelasticity of conductors—dissipative case including only thermal dissipation—play a paradigmatic role in the uncovery of the formal structure of a theory of continua and establishing the most striking general results of the theories of fracture and of progress of phase-transition fronts; (ii) this, contrary to a too much superficial common feeling, is facilitated by considering first the *exact* case of finite deformations and fields; (iii) the latter argument is of necessity developed by clearly distinguishing between the *actual configuration* of a body and a *reference configuration*; and (iv) one must also distinguish between field equations—there exists one such equation (say, the Euler–Lagrange equation for the non-dissipative case) for each independent field in the theory—and canonical balance laws, which pertain to the whole physical system and reflect the invariance—or lack of invariance—of the system under fundamental transformations. Examples of such canonical balance laws are the energy balance and the much less known and exploited balance of canonical momentum—what we called balance of 'pseudo-momentum'. Just like the former that is a scalar equation, the balance of pseudo-momentum or, for materially inhomogeneous bodies—as we like to say—the unbalance of pseudo-momentum, is that material covectorial equation whose global form and constituents, material forces, are the fundamental ingredients of the theory of driving forces on material defects, macroscopic cracks being considered as such defects in their own right. In so far as the propagation of phase-transition fronts is concerned, it is the jump relation associated with the lack of strict conservation of pseudo-momentum which does govern the progress of the front. Note in passing that energy and pseudo-momentum are the time-like and space-like components of a unique four-dimensional geometric object. The extraordinary power of these canonical considerations on the material manifold is best exemplified in the case where the material body exhibits several degress of freedom, the additional degrees of freedom relating either to an internal mechanical structure (such as in liquid crystals) or to an internal electromagnetic structure (such as in elastic ferromagnets).

The nature of material forces versus physical ones in liquid crystals described by the Frank–Leslie–Ericksen theory (so-called director theory) was thus carefully examined in Maugin & Trimarco (1995*a*), while the application to elastic ferromagnets with a magnetic (spin) substructure was dealt with in details by Maugin (1992) for soliton propagation and in Fomethe & Maugin (1996, 1997, 1998) for the study of driving forces on cracks and phase-transition fronts. The problem of the application of such concepts to polar elastic bodies then is a natural step forward, especially inasmuch as questions of fracture and phase-transition fronts will necessary appear in such materials, e.g. polar crystals such as KNO_3 or $NaNO_2$ (see Askar 1986; Pouget & Maugin 1989). Pioneering works on a specific aspect of this, without connection with the present general approach, are due to Jaric (1978) and Vukobrat (1989), who formally examined the question of the formulation of conservation laws and path-independent J integrals in the linear elasticity of so-called micropolar media (Eringen 1968). However, in the light of remarks formulated above, we should better first consider the general framework of finite strains. Thus we shall base our considerations on what appears to be, so far, the best formulation of finite strain polar elasticity, the Kafadar–Eringen formulation of 1971—reviewed in Eringen & Kafadar

(1976). This formulation is best suited for the following reasons. First, it does base the energy considerations on local changes of objects and not the objects themselves. By this we mean that, in so far as the microstructure is concerned, the energy is a function of the orthogonal transformations of the microstructure and *not* of vector fields (directors) representing directly the microstructure (for the foundations of the director approach see Truesdell & Noll (1965) and Naghdi (1972)). This is a sound physical description as directors themselves have no energetic features *a priori* (but their gradients *do*). Second, it allows directly a material formulation in which time and material coordinates enter automatically with their simply commuting derivatives. Finally, it is that formulation which has best considered invariance matters by introducing correctly the strain measures called the Cosserat and wryness tensors and consequently affords the best available framework for studying the notions of material uniformity (in the sense of W. Noll, C. C. Wang & M. Epstein), inhomogeneity and material symmetry . The accompanying proof of inclusive results is paradoxically much simpler than in the linearized theory of small strains and small rotations! As the inertial term attributed to the microstructure may sometimes create difficulties which we do not consider in view of our applications, the following presentation will be essentially confined to the case of quasi-statics, remarks concerning inertial terms being given only for the sake of curiosity.

Following the logic transparent in the introduction given above, the main body of the text is developed in the following way. Section 2 presents the main notation and 'field equations', while 'conservation laws' are formulated in two different ways in § 3, either by strict algebraic manipulations or through the application of Noether's theorem. The notions of driving force, path-independent integral and energy-release rate useful in fracture studies are expanded in § 4. Elements of full dynamics are given in § 5 with the introduction of 'pseudo-momentum' *per se*. Remarks on time and length scales are given at that point. The case of thermoelasticity of conductors is looked upon in § 6. This helps one formulate the problem of the progress of phase-transition fronts in § 7. The approximation of small strains and small rotations in § 8 permits the confrontation with simple approaches. Section 9 is conclusive. The intrinsic dyadic notation is used throughout, but many expressions are also given in tensorial-index notation when a risk of confusion arises. Some intermediate proofs are more easily carried in the latter for those readers not so familiar with manipulations using the intrinsic notation.

2. Field equations

(*a*) *Elements of kinematics*

Let $\boldsymbol{x}$, with Cartesian components x^k, $k = 1, 2, 3$, denote the actual placement in physical Euclidean space $\mathbb{E}^3$ at time t of the material point $\boldsymbol{X}$ (referred to a material basis $\boldsymbol{G}_K$, $K = 1, 2, 3$, on the material manifold $\mathcal{M}^3$). The point $\boldsymbol{x}$ belongs to the actual configuration $\mathcal{K}_t$, while $\boldsymbol{X}$ is referred to a reference configuration $\mathcal{K}_{\mathrm{R}}$. Assuming all functional behaviours sufficiently smooth for the moment, the motion of $\boldsymbol{X}$ in $\mathbb{E}^3$ is described by the time-parametrized diffeomorphism $\chi(\boldsymbol{X}, t)$ while its regular inverse is noted $\chi^{-1}(\boldsymbol{x}, t)$. That is,

$$\boldsymbol{x} = \chi(\boldsymbol{X}, t), \quad \boldsymbol{X} = \chi^{-1}(\boldsymbol{x}, t), \tag{2.1}$$

with (T denoting transpose)

$$\boldsymbol{F} := \left.\frac{\partial \chi}{\partial \boldsymbol{X}}\right|_t = (\nabla_{\mathrm{R}}\chi)^{\mathrm{T}}, \quad J := \det \boldsymbol{F} > 0 \quad \text{always} \tag{2.2}$$

and

$$\boldsymbol{F}^{-1} := \left.\frac{\partial \chi^{-1}}{\partial \boldsymbol{x}}\right|_t = (\nabla_{\mathrm{R}}\chi^{-1})^{\mathrm{T}}, \quad \boldsymbol{F}\cdot\boldsymbol{F}^{-1} = \mathbf{1}, \quad \boldsymbol{F}^{-1}\cdot\boldsymbol{F} = \mathbf{1}_{\mathrm{R}}, \tag{2.3}$$

where $\mathbf{1}$ and $\mathbf{1}_{\mathrm{R}}$ are unit dyadics in $\mathcal{K}_t$ and $\mathcal{K}_{\mathrm{R}}$, respectively. The classical finite Cauchy strain is defined by

$$\boldsymbol{C} = \boldsymbol{F}^{\mathrm{T}}\boldsymbol{F} = \boldsymbol{C}^{\mathrm{T}}, \quad \text{i.e. } C_{KL} = \delta_{ij}F^i_{\cdot K}F^j_{\cdot L}. \tag{2.4}$$

Accordingly, the components of $\boldsymbol{F}^{-1}$ are denoted by $(\boldsymbol{F}^{-1})^K_{\cdot i}$. However, it should be remarked that while $\mathbb{E}^3$ is neutral from the tensorial viewpoint (i.e. being Euclidean, it does not distinguish between covariant and contravariant objects), $\mathcal{M}^3$ does make that difference so that in general, in tensorial notation, the up or down place of upper case Latin indices is important, while that of lower case Latin indices is irrelevant. Hence (2.4) may also be written $C_{KL} = F^i_{\cdot K}F^i_{\cdot L} = F^i_{\cdot K}F_{iL}$. The inverse of $\boldsymbol{C}$, $\boldsymbol{C}^{-1}$, has thus contravariant material components and, by transposition, $\boldsymbol{F}^{\mathrm{T}}$ should have components $(\boldsymbol{F})^{\cdot i}_K$. In parallel to (2.2) we can define the partial time derivative of χ as being the physical velocity $\boldsymbol{v}$:

$$\boldsymbol{v} := \left.\frac{\partial \chi(\boldsymbol{X},t)}{\partial t}\right|_{\boldsymbol{X}} = \dot{\chi}. \tag{2.5}$$

Correspondingly, we have the material velocity given intrinsically by

$$\boldsymbol{V} := \left.\frac{\partial \chi^{-1}(\boldsymbol{x},t)}{\partial t}\right|_{\boldsymbol{x}} = -\boldsymbol{F}^{-1}\cdot\boldsymbol{v}, \tag{2.6}$$

where the last relationship is checked by use of the chain rule of differentiation.

Let $\boldsymbol{A}(\boldsymbol{X},t)$ be a tensor-valued function of $(\boldsymbol{X},t)$, of which we do not specify the order, and $f(\boldsymbol{A}(\boldsymbol{X},t);\boldsymbol{X},t)$ be a scalar-valued function of $\boldsymbol{A}$ and an explicit function of $\boldsymbol{X}$ and t. Then we note that the total material gradient and material-time derivative of f are given by

$$\nabla_{\mathrm{R}} f = \frac{\partial f}{\partial \boldsymbol{X}} = \mathrm{tr}\left(\frac{\partial f}{\partial \boldsymbol{A}}\cdot(\nabla_{\mathrm{R}}\boldsymbol{A})^{\mathrm{T}}\right) + \left(\frac{\partial f}{\partial \boldsymbol{X}}\right)_{\mathrm{expl}} \tag{2.7}$$

and

$$\dot{f} \equiv \left.\frac{\partial f}{\partial t}\right|_{\boldsymbol{X}} = \mathrm{tr}\left(\frac{\partial f}{\partial \boldsymbol{A}}\cdot\left.\frac{\partial \boldsymbol{A}}{\partial t}\right|_{\boldsymbol{X}}\right) + \left(\frac{\partial f}{\partial t}\right)_{\mathrm{expl}}, \tag{2.8}$$

where the *explicit* material gradient and partial time derivative are obviously taken at fixed $\boldsymbol{A}$ and t, and fixed $\boldsymbol{A}$ and $\boldsymbol{X}$, respectively. For functions $f(\boldsymbol{X},t)$, material gradient and material time derivative commute. For functions $g(\boldsymbol{x},t)$, it is the ∇ gradient and the partial time derivative at fixed $\boldsymbol{x}$ that commute. These commutations greatly facilitate the subsequent manipulations.

Now equations (2.1) describe the classical motion of a material point, as the latter is commonly viewed in classical continuum mechanics—this point has 'no structure'. In polar materials, also called Cosserat continua or oriented continua, one endowes each material point $\boldsymbol{X}$ with a structure that can only rotate . This additional degree

of freedom, adjoined to χ to give the generalized motion of the continuum, is thus best described by an orthogonal transformation $\mathfrak{X}$ which maps a unit vector $\boldsymbol{D}$ of $\mathcal{K}_{\mathrm{R}}$ at $\boldsymbol{X}$ into a unit vector $\boldsymbol{d}$ of $\mathcal{K}_t$ at $\boldsymbol{x}$, while $\boldsymbol{F}$ mapped a small increment $\mathrm{d}\boldsymbol{X}$ into a small increment $\mathrm{d}\boldsymbol{x}$. Thus $\mathfrak{X}$ enjoys the following properties:

$$\mathfrak{X} = \{\mathfrak{X}^i_{\cdot K}\}, \tag{2.9 a}$$

$$\mathfrak{X}^{-1} = \{\mathfrak{X}^K_{\cdot i}\} = \mathfrak{X}^{\mathrm{T}}, \tag{2.9 b}$$

$$\det \mathfrak{X} = +1, \tag{2.9 c}$$

$$\mathfrak{X} \cdot \mathfrak{X}^{-1} = \mathfrak{X} \cdot \mathfrak{X}^{-1} = \mathbf{1}, \tag{2.9 d}$$

$$\mathfrak{X}^{-1} \cdot \mathfrak{X} = \mathfrak{X}^{\mathrm{T}} \cdot \mathfrak{X} = \mathbf{1}_{\mathrm{R}}. \tag{2.9 e}$$

Following Kafadar & Eringen (1971), we introduce the so-called Cosserat and wryness tensors, respectively, $\mathfrak{E}$ (German gothic E) and $\mathfrak{G}$ (German gothic G), both material measures of finite strains, by

$$\mathfrak{E} := \boldsymbol{F}^{\mathrm{T}} \cdot \mathfrak{X}, \tag{2.10 a}$$

$$\mathfrak{G} := \tfrac{1}{2}\mathfrak{X}^{-1}\dot{\times}(\nabla_{\mathrm{R}}\mathfrak{X}), \tag{2.10 b}$$

where the symbolism introduced is such that $\mathfrak{E}$ is the material pull back of $\mathfrak{X}$ and $\mathfrak{G}$ is a material geometric object which is axial on its first index, being associated by duality with the skew quantity obtained by taking the material gradient of (2.9). In components these material covariant objects read (note that in (2.10) the vector product $\times$ is effected on $\mathcal{M}^3$ while the (superimposed) inner product is effected in $\mathbb{E}^3$)

$$\mathfrak{E}_{KL} = \delta_{ij} F^i_{\cdot K} \mathfrak{X}^j_{\cdot L}, \quad \mathfrak{G}_{QL} = \tfrac{1}{2}\varepsilon^{\cdot\cdot P}_{QK} \mathfrak{X}^K_{\cdot i} \mathfrak{X}^i_{\cdot P,L}. \tag{2.11}$$

It is more than a curiosity to note that

$$\boldsymbol{C} = \mathfrak{C} \cdot \mathfrak{C}^{\mathrm{T}}. \tag{2.12}$$

The (physical) velocity associated with $\mathfrak{X}$ is obviously defined and given by

$$\dot{\mathfrak{X}} := \left.\frac{\partial \mathfrak{X}}{\partial t}\right|_{\boldsymbol{X}} = \bar{\nu} \cdot \mathfrak{X}, \quad \bar{\nu} = \dot{\mathfrak{X}} \cdot \mathfrak{X}^{-1} = -\bar{\nu}^{\mathrm{T}}. \tag{2.13}$$

The axial vector ν associated with the skew tensor velocity $\bar{\nu}$ is given by

$$\nu = -\tfrac{1}{2}\,\mathrm{dual}\,\bar{\nu}, \quad \text{i.e. } \nu_k = -\tfrac{1}{2}\varepsilon^{\cdot\cdot q}_{kp}\bar{\nu}^p_{\cdot q}. \tag{2.14}$$

Reciprocally,

$$\bar{\nu} = -\,\mathrm{dual}\,\nu, \quad \text{i.e. } \bar{\nu}^{km} = -\varepsilon^{kmn}\nu_n. \tag{2.15}$$

Again, the skew symmetry inherent in ν is a result of the purely rotational character of the microstructure. In conclusion of these kinematical developments, the direct generalized motion of our polar continuum is given by the nine functions

$$\{\chi(\boldsymbol{X}, t), \mathfrak{X}(\boldsymbol{X}, t)\}, \tag{2.16}$$

with velocities $\{\boldsymbol{v}, \dot{\mathfrak{X}}\}$, first material gradients (deformations) $\{\boldsymbol{F}, \nabla_{\mathrm{R}}\mathfrak{X}\}$ and independent finite strains $\{\mathfrak{E}, \mathfrak{G}\}$ in the so-called direct motion description (Maugin 1992). Equation (2.16) clearly defines the basic fields in terms of the space-time parametrization $(\boldsymbol{X}, t)$ of this description. We are thus unequivocally equipped for a field approach to polar elastic media.

(*b*) *Lagrange equations of motion*

For a general theory of polar elastic media of the first (gradient) order, in all generality we should start with a Lagrangian function L per unit volume of the reference configuration as

$$L = \bar{L}(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{F}, \mathfrak{X}, \dot{\mathfrak{X}}, \nabla_{\mathrm{R}}\mathfrak{X}; \boldsymbol{X}, t), \tag{2.17}$$

where the field quantities and derivatives are to the left of the semi-colon and the parameters of the description to the right. Homogeneity of physical space rules out any explicit dependence on $\boldsymbol{x}$ and the lack of dissipation rules out the explicit dependence on t. That is,

$$\left(\frac{\partial \bar{L}}{\partial \boldsymbol{x}}\right)_{\mathrm{expl}} = \boldsymbol{0}, \tag{2.18\,a}$$

$$\left(\frac{\partial \bar{L}}{\partial t}\right)_{\mathrm{expl}} = 0. \tag{2.18\,b}$$

In quasi-statics, where inertial effects are altogether discarded, the Lagrangian function per unit reference volume reduces to minus the elastic potential W. Finally, nothing *a priori* rules out a possible dependence on $\boldsymbol{X}$, so that we have

$$\frac{\partial \bar{L}}{\partial \boldsymbol{v}} = \boldsymbol{0}, \quad \frac{\partial \bar{L}}{\partial \dot{\mathfrak{X}}} = \boldsymbol{0}, \quad \bar{L} = -\bar{W}, \tag{2.19}$$

with

$$W = \bar{W}(\boldsymbol{F}, \mathfrak{X}, \nabla_{\mathrm{R}}\mathfrak{X}; \boldsymbol{X}), \tag{2.20}$$

which may be considered as Lagrangian function for all practical purposes. Accordingly (see, for instance, Maugin (1970) in a more complex case accounting also for a finite spatial extension of the material body and boundary conditions), in the absence of external stimuli and identifying easily generalized coordinates and momenta, the basic field equations are given by the Euler–Lagrange equations associated to the expression (2.20). That is, at all regular material points $\boldsymbol{X}$ in the material body,

$$\mathcal{E}_{\chi} := \frac{\partial \bar{L}}{\partial \boldsymbol{x}} - \nabla_{\mathrm{R}} \cdot \frac{\partial \bar{L}}{\partial (\nabla_{\mathrm{R}}\chi)} = \div_{\mathrm{R}} \boldsymbol{T} = \boldsymbol{0} \tag{2.21}$$

and

$$\mathcal{E}_{\mathfrak{X}} := \frac{\partial \bar{L}}{\partial \mathfrak{X}} - \nabla_{\mathrm{R}} \cdot \frac{\partial \bar{L}}{\partial (\nabla_{\mathrm{R}}\mathfrak{X})} = \div_{\mathrm{R}} \mathcal{M} + \mathcal{N} = \boldsymbol{0}, \tag{2.22}$$

in which we have defined the two-point tensor fields $\boldsymbol{T}$, $\mathfrak{M}$ and $\mathfrak{N}$, with tensorial components $T^K_{\cdot i}$ or T^{Ki}, $\mathfrak{M}^{KL}_{\cdot\cdot i}$ or $\mathfrak{M}^{KLi}$ and $\mathfrak{N}^K_{\cdot i}$ or $\mathfrak{N}^{Ki}$, respectively,

$$\boldsymbol{T} := \left(\frac{\partial \bar{W}}{\partial \boldsymbol{F}}\right)^{\mathrm{T}}, \tag{2.23\,a}$$

$$\mathfrak{M} := \left(\frac{\partial \bar{W}}{\partial (\nabla_{\mathrm{R}}\mathfrak{X})}\right)^{\mathrm{T}}, \tag{2.23\,b}$$

$$\mathfrak{N} := -\left(\frac{\partial \bar{W}}{\partial \mathfrak{X}}\right)^{\mathrm{T}}, \tag{2.23\,c}$$

where the first is none other than the first Piola–Kirchhoff stress and the second

and third may be referred to as the Piola–Kirchhoff microstress and the microforce, respectively. Here the divergence $\div_{\mathrm{R}}$ (not a covariant one; it just uses partial derivatives $\partial/\partial X^K$) is taken with respect to the first index of tensors. To be complete we also introduce the other derivative, a material covector, $\boldsymbol{f}^{\mathrm{inh}}$ by

$$\boldsymbol{f}^{\mathrm{inh}} := -\left(\frac{\partial \bar{W}}{\partial \boldsymbol{X}}\right)_{\mathrm{expl}}, \tag{2.24}$$

which for the moment serves no purpose. If we recognize in (2.21) the balance of physical momentum in its quasi-static form in the absence of body force, we must admit that (2.22), although indeed the field equation associated with the microstructure, *a priori* does not ring a bell. The reason for that is that, if (2.21) reflects, via the invariance (2.18) (sometimes referred to as homogeneity of space by physicists), the balance of physical momentum, albeit in a particular case, (2.22) is not *a priori* related to any classical conservation law of mechanics. Indeed, the second classical invariance in mechanics is that under time-independent rotation in the physical framework . This is the rotational part (also referred to as isotropy of space by physicists) of what may be called Euclidean invariance . As the group of proper rotations $SO(3)$ is connected, it is sufficient to study the invariance under infinitesimal generators (skewsymmetric objects). Applying this to the scalar function $\bar{W}$ yields the following constraint (in fact three independent scalar equations (see Maugin 1970)) (skew means taking the skewsymmetric part of the corresponding spatial tensor):

$$(\boldsymbol{F}\cdot\boldsymbol{T} - \mathfrak{X}\cdot\mathcal{N} + \nabla_{\mathrm{R}}\mathfrak{X} : \mathfrak{M})_{\mathrm{skew}} = \boldsymbol{0}. \tag{2.25}$$

The very form of this expression suggests to take the material inner product of equation (2.22) with $\bar{\nu}$, take the skew part of the resulting spatial tensor equation, i.e.

$$\mathcal{C}_{\mathfrak{X}} := \{\mathfrak{X}\cdot(\div_{\mathrm{R}}\mathfrak{M} + \mathfrak{N})\}_{\mathrm{skew}} = \boldsymbol{0}, \tag{2.26}$$

and combine the result with (2.25) to reach the following sensible equation:

$$\div_{\mathrm{R}}\boldsymbol{M} + \boldsymbol{F} \mathbin{\dot{\times}} \boldsymbol{T} = \boldsymbol{0}, \tag{2.27}$$

where the symbolism $\mathbin{\dot{\times}}$ has the opposite operational meaning to the symbolism $\dot{\times}$ in (2.10 *b*), meaning vector product in $\mathbb{E}^3$ and (subimposed) inner product on $\mathcal{M}^3$ (see index notation in equation (2.29) below), and the two-point tensor field $\boldsymbol{M}$ of tensorial components $M^K i$ is defined by

$$\boldsymbol{M} := \{M^K i = \varepsilon^{ipq}\mathfrak{M}^{KL}_{\cdot\cdot p}\mathfrak{X}_{qL}\}. \tag{2.28}$$

Equation (2.27) indeed is the balance of physical moment of momentum , in quasi-static form. In particular, whenever W does not depend on the microstructure, this equation reduces to the classical one

$$\boldsymbol{F} \mathbin{\dot{\times}} \boldsymbol{T} = \boldsymbol{0} \quad \text{or} \quad \varepsilon_{ijk}F^j_{\cdot K}T^{Kk} = 0 \quad \text{or} \quad (\boldsymbol{F}\cdot\boldsymbol{T})_{\mathrm{skew}} = \boldsymbol{0}, \tag{2.29}$$

which reflects the symmetry of the Cauchy stress—the stress tensor in $\mathcal{K}_t$—in the absence of microstructure, spin and body couple.

Obviously then, in field theory, equation (2.27) is not a direct field equation but the result of a manipulation that takes the 'isotropy of space' into account. But we have not yet exhausted the possible conservation laws as we said practically nothing about (2.18 *b*) and (2.24).

3. Canonical balance laws

Canonical balance laws are those balance laws which concern the whole physical system under consideration and not one degree of freedom in particular. They in fact relate to the space-time parametrization of the problem, hence here the invariance or lack of invariance under changes of space-time parameters $\boldsymbol{X}$ and t. Two essential methods can be used to reach those equations. The sophisticated one makes use of the celebrated Noether's theorem according to which a *conservation law is associated with each parameter of the description* (see Maugin 1992; Maugin & Trimarco 1992; Soper 1976). The 'poor man' one uses algebraic manipulations of the already known field equations. We naively start with the latter. To that purpose we add up the inner product of equation (2.21) with $\boldsymbol{v}$ to the contracted product of equation (2.22) with $\dot{\mathfrak{X}}$, i.e. we form the vanishing scalar quantity

$$\boldsymbol{v}\cdot\mathcal{E}_\chi+\mathrm{tr}(\dot{\mathfrak{X}}\cdot\mathcal{E}_{\mathfrak{X}})=0. \tag{3.1}$$

This, by integration by parts and use of commutation rules between derivatives and of the obvious identity

$$\mathrm{tr}(\bar{\nu}\cdot\mathcal{C}_{\mathfrak{X}})=0, \tag{3.2}$$

where $\mathcal{C}_{\mathfrak{X}}$ is the vanishing skew tensor defined in (2.26), yields, after some algebra that we leave to the reader,

$$\div_{\mathrm{R}}(\boldsymbol{T}\cdot\boldsymbol{v}+\boldsymbol{M}\cdot\nu)-\left.\frac{\partial W}{\partial t}\right|_{\boldsymbol{X}}=0. \tag{3.3}$$

On account of the second equation of (2.18), this equation is the local statement of energy conservation in quasi-statics. It is readily verified that this can also be written in terms of the original field $\mathfrak{X}$ and its time derivative as

$$\div_{\mathrm{R}}(\boldsymbol{T}\cdot\dot{\chi}+\mathfrak{M}\cdot\dot{\mathfrak{X}})-\left.\frac{\partial W}{\partial t}\right|_{\boldsymbol{X}}=0, \tag{3.4}$$

because there holds the trivial identity $\boldsymbol{M}\cdot\nu\equiv\mathfrak{M}\cdot\dot{\mathfrak{X}}$.

We can perform the parallel space-like operation on equations (2.21) and (2.22) by forming the following vanishing material covector (cf. equation (3.1)):

$$\boldsymbol{F}^{\mathrm{T}}\cdot\mathcal{E}_\chi+(\nabla_{\mathrm{R}}\mathfrak{X})\cdot\mathcal{E}_{\mathfrak{X}}\equiv\boldsymbol{0}. \tag{3.5}$$

Integration by parts, exploitation of the commutation rules between various derivatives and taking account of the constitutive equations (2.23) and of the definition (2.24) results in the following fully material balance law:

$$\div_{\mathrm{R}}\boldsymbol{b}+\boldsymbol{f}^{\mathrm{inh}}=\boldsymbol{0}, \tag{3.6}$$

where the mixed material stress tensor $\boldsymbol{b}$ is canonically defined by

$$\boldsymbol{b}=W\boldsymbol{1}_{\mathrm{R}}-\boldsymbol{T}\cdot\boldsymbol{F}-\mathfrak{M}:\nabla_{\mathrm{R}}\mathfrak{X}. \tag{3.7}$$

This is the polar-elasticity generalization of the so-called Eshelby stress tensor (coinage by Maugin & Trimarco in 1992—originally this would have been referred to as the energy-momentum tensor by Eshelby (see Maugin 1993), but this denomination is unfortunately misleading). The presence of the material inhomogeneity force $\boldsymbol{f}^{\mathrm{inh}}$—according to which the material properties of the solid depend on the material point $\boldsymbol{X}$, here supposedly in a smooth way—in equation (3.6), obviously shows that

equation (3.6) is the result of the lack of invariance of the physical system under translations in $\boldsymbol{X}$ space. This is indeed material inhomogeneity in its primitive definition. The full dynamics form of equation (3.7) is referred to by us as the equation of pseudo-momentum or material momentum (in a general setting, i.e. even when dissipative processes are present) (see below). It is the balance of canonical momentum in the non-dissipative variational formulation à la Hamilton–Lagrange. In the absence of elastic inhomogeneities, equation (3.6) reduces to the following mere identity (as this is not independent of equation (2.21)):

$$\div_{\mathrm{R}} \boldsymbol{b} = \boldsymbol{0}. \tag{3.8}$$

Three important questions can be raised concerning (3.6) and (3.8). First, the way they were obtained here, in fact, is a polar-elasticity, materially inhomogeneous, generalization of an identity initially pointed out by Ericksen (1977) in classical elastostatics as the original Ericksen identity would simply read $\boldsymbol{F}^{\mathrm{T}} \cdot \mathcal{E}_{\chi} \equiv \boldsymbol{0}$ in our formalism. This was generalized to the case of non-dissipative liquid crystals—exhibiting thus an internal degree of freedom—by Maugin & Trimarco (1995*a*). Here, the generalization is equation (3.5).

Second, equations (3.3) and (3.6) should be *compatible* with one another in the sense that in a good canonical mechanics (3.3) and (3.6) are but the time-like and space-like components of a four-dimensional balance law. This is indeed the case as we let the reader check that (3.3) is none other than the equation

$$\boldsymbol{V} \cdot (\div_{\mathrm{R}} \boldsymbol{b} + \boldsymbol{f}^{\mathrm{inh}}) = 0, \tag{3.9}$$

where $\boldsymbol{V}$ is the material velocity defined in (2.6).

Third, one may wonder what is the *symmetry* condition on $\boldsymbol{b}$ so that the original law of moment of momentum is verified. In classical elastostatics, it was shown by Epstein & Maugin (1990) that this condition, on account of (2.29), reads

$$(\boldsymbol{C} \cdot \boldsymbol{b})_{\mathrm{skew}} = -\boldsymbol{F}^{\mathrm{T}} \cdot (\boldsymbol{F} \cdot \boldsymbol{T})_{\mathrm{skew}} \cdot \boldsymbol{F} \equiv \boldsymbol{0}, \tag{3.10}$$

or

$$\boldsymbol{C} \cdot \boldsymbol{b} = \boldsymbol{b}^{\mathrm{T}} \cdot \boldsymbol{C}; \tag{3.11}$$

expressis verbis: '$\boldsymbol{b}$ is symmetric with respect to the Cauchy finite strain', which thus plays the role of deformed metric. In the present case, this requires a little bit more work because (3.10), on account of (2.25), is now replaced by the following covariant material condition:

$$\div_{\mathrm{R}} \mathfrak{B} + \underline{\mathcal{C}} = (\boldsymbol{C} \cdot \boldsymbol{b})_{\mathrm{skew}}, \tag{3.12}$$

in which the fully material tensors $\mathfrak{B}$ and $\underline{\mathcal{C}}$ may tentatively be called the Eshelby hyperstress tensor $\mathfrak{B}$ and the material couple tensor, respectively. In components these are defined by (square brackets indicate skewsymmetrization)

$$\mathfrak{B}^{K}_{\cdot PQ} \equiv \mathfrak{B}^{K}_{\cdot [PQ]} = \mathfrak{C}_{[P|L|} \mathcal{M}^{LK}_{\cdot\cdot i} F^{i}_{\cdot Q]} \tag{3.13\,a}$$

and

$$\mathcal{C}_{PQ} \equiv \mathcal{C}_{[PQ]} = -(\mathfrak{C}_{[P|K|} F^{i}_{\cdot Q]})_{,L} \mathcal{M}^{LK}_{\cdot\cdot i}. \tag{3.13\,b}$$

The proof of (3.12) is as follows. Compute $(\boldsymbol{C} \cdot \boldsymbol{b})_{\mathrm{skew}}$ from the definitions (2.4) and (3.7) of $\boldsymbol{C}$ and $\boldsymbol{b}$. In the obtained expression replace $(\boldsymbol{F} \cdot \boldsymbol{T})_{\mathrm{skew}}$ by its value extracted from (2.25), integrate by parts, introduce $\mathfrak{E}$ from its definition (2.10) and enforce (2.22), so that one is left with (3.12) on account of definitions (3.13). An

equation such as (3.12), an *identity*, is entirely new. It is of interest because it nicely completes the mechanics on the material manifold—or Eshelbian mechanics that we developed since 1989. In particular, equation (3.12) gives a hint of what may occur in the full material transcription of continuum theories with microstructure. This is in agreement with the recently developed equations of the theory of dislocated materials with torsion (Kröner & Maugin 1998), where an equation with the same structure as (3.12) of necessity appears—in that case the structure is not related to a rigid microstructure, but to anelasticity.

We now have apparently exhausted the balance laws, whether canonical or not, of the present theory as we finally considered the invariance under changes of both the fields $(\chi, \mathfrak{X})$ and the parameters $(\boldsymbol{X}, t)$. This is confirmed by the application of Noether's theorem (the sophisticated way), for instance, as mathematically stated in Maugin (1993) or Soper (1976). Indeed, direct application of the formulas given in these books or of formulas (5.3)–(5.4) in Maugin & Trimarco (1995*a*) yields equations (3.4) and (3.6) at once. However, we must remark that both physical balance of moment of momentum—here expressed by equation (2.27)—and its fully material transcription (3.12) are not directly obtained; they always are the result of further manipulations. Worse than this, had we considered the moment-of-momentum form (2.27) to start with, instead of the original field equation (2.22), we would have met insuperable obstacles to prove equation (3.6). If we follow along the lines of our book (Maugin 1993, ch. 4), we could also construct new material balance laws (not necessarily conservation laws) corresponding to the scalar moment and vectorial moment of equation (3.6), i.e. symbolically and in accord with basic notions of quaternion algebra,

$$\boldsymbol{X} \cdot (\div_{\mathrm{R}} \boldsymbol{b} + \boldsymbol{f}^{\mathrm{inh}}) = 0 \tag{3.14}$$

and ($\otimes$ indicates the tensor product)

$$\{\boldsymbol{X} \cdot \boldsymbol{C} \otimes (\div_{\mathrm{R}} \boldsymbol{b} + \boldsymbol{f}^{\mathrm{inh}})\}_{\mathrm{skew}} = \boldsymbol{0}, \tag{3.15}$$

where $\boldsymbol{C}$ is used as the deformed material metric in order to take the skew part of a truly covariant object. The computation of the left-hand side of (3.15) provides an expression for $(\boldsymbol{C} \cdot \boldsymbol{b})_{\mathrm{skew}}$ and the elimination of this skew quantity between this expression and (3.12) yields the following quasi-static balance of moment of pseudo-momentum:

$$\div_{\mathrm{R}} \bar{\mathfrak{B}} + \bar{\underline{\mathcal{C}}} + \underline{\mathcal{C}}^{\mathrm{inh}} = \boldsymbol{0}, \tag{3.16}$$

wherein both $\bar{\mathfrak{B}}$ and $\bar{\underline{\mathcal{C}}}$ have orbital contributions and, together with $\underline{\mathcal{C}}^{\mathrm{inh}}$, are defined, component wise, by

$$\begin{aligned} \bar{\mathfrak{B}}^{P}_{\cdot LQ} &\equiv \bar{\mathfrak{B}}^{P}_{\cdot [LQ]} := \mathfrak{B}^{P}_{\cdot LQ} + X^{K} C_{K[L} b^{P}_{\cdot Q]}, \\ \bar{\mathcal{C}}_{LQ} &\equiv \bar{\mathcal{C}}_{[LQ]} := \mathcal{C}_{LQ} + X^{K} C_{K[L,|P|} b^{P}_{\cdot Q]}, \\ \mathcal{C}^{\mathrm{inh}}_{LQ} &:= -X^{K} C_{K[L} f^{\mathrm{inh}}_{Q]}. \end{aligned} \tag{3.17}$$

We shall not dwell in greater detail with (3.14), (3.15) and (3.16) as they do not seem to play, for the moment, a great role in further applications (but (3.14) in classical elasticity is known to be related to the spherical expansion of voids or inclusions and (3.15) to the progress of defects of the disclination type (cf. Maugin 1993)). Finally, noting that equation (2.25) represents a set of three first-order partial differential equations which can be integrated along characteristics, a first integral of (2.25) is

readily obtained as

$$W = \hat{W}(\mathfrak{E}, \mathfrak{G}; \boldsymbol{X}), \tag{3.18}$$

where $\mathfrak{E}$ and $\mathfrak{G}$ are the Cosserat and wryness tensors defined in (2.10). The functional dependence (3.18) is due to Kafadar & Eringen (1971). It is objective, i.e. form invariant, under time-dependent rotations of the actual frame (in $\mathcal{K}_t$). The fact that such a form results from the rotational invariance condition (2.25)—in which no time was involved—is connected with the non-dissipative nature of the material considered (its behaviour involves neither time derivatives nor histories of the state variables). In other words, the rotational part of Euclidean invariance and objectivity yields the same restriction in this case. Thus (2.25) is automatically fulfilled and the constitutive equations (2.23 *a*) and (2.28) are now given by

$$T^{Ki} = \frac{\partial \hat{W}}{\partial \mathfrak{E}_{KL}} \mathfrak{X}^i_{\cdot L}, \quad M^{Ki} = \frac{\partial \hat{W}}{\partial \mathfrak{G}_{LK}} \mathfrak{X}^i_{\cdot L}. \tag{3.19}$$

The beauty and symmetry of these formulae must be underlined. It should also be noted for all practical purposes that in computing $\partial W/\partial \mathfrak{X}^i_{\cdot P}$ one should account for the dependence of $\hat{W}$ on $\mathfrak{X}$ via both $\mathfrak{E}$ and $\mathfrak{G}$—the latter through $\mathfrak{X}^{-1}$ since one can show the following useful results which are consequences of equations (2.9d) and (2.9 *e*):

$$\frac{\partial (\mathfrak{X}^{-1})^L_{\cdot j}}{\partial \mathfrak{X}^i_{\cdot K}} = -(\mathfrak{X}^{-1})^K_{\cdot j} (\mathfrak{X}^{-1})^L_{\cdot i}, \quad \frac{\partial \mathfrak{X}^i_{\cdot L}}{\partial (\mathfrak{X}^{-1})^K_{\cdot j}} = -\mathfrak{X}^i_{\cdot K} \mathfrak{X}^j_{\cdot L}, \tag{3.20}$$

from which it follows that

$$\frac{\partial \mathfrak{G}_{MN}}{\partial (\mathfrak{X}^{-1})^K_{\cdot j}} = \tfrac{1}{2} \varepsilon^P_{MK} \mathfrak{X}^j_{\cdot P,N}. \tag{3.21}$$

These intermediate computations are essential in checking the balance laws enunciated above directly from (3.16).

4. Fracture and energy-release rate

It is understood that the balance equations constructed above, whether canonical or not, are valid at all material points where the fields do not present singularities. For a finite material region V made of such points, we can integrate these balance equations over V and use the standard divergence theorem and commutation rule between material integration and material time derivative. For instance, from equations (2.21) and (2.22) contracted, respectively, by virtual velocity fields $\boldsymbol{v}^*$ and $\dot{\mathfrak{X}}^*$, and adding up the results, we can write a weak form of the balance of physical forces and microforces as

$$\int_{\partial V} (\mathcal{T} \cdot \boldsymbol{v}^* + M \cdot \dot{\mathfrak{X}}^*) \, \mathrm{d}S - \int_V \mathrm{tr}\{\boldsymbol{T} \cdot (\nabla_{\mathrm{R}} \boldsymbol{v}^*)^{\mathrm{T}} + \mathcal{M} \cdot (\nabla_{\mathrm{R}} \dot{\mathfrak{X}}^*)^{\mathrm{T}}\} \, \mathrm{d}V = 0, \tag{4.1}$$

where we set (either these are so-called natural boundary conditions with prescribed loads $\mathcal{T}$ and M or these are mere definitions of the right-hand sides)

$$\boldsymbol{N} \cdot \boldsymbol{T} = \mathcal{T}, \quad \boldsymbol{N} \cdot \mathcal{M} = M, \tag{4.2}$$

where $\boldsymbol{N}$ is the unit outward normal to the supposedly regular surface ∂V bounding the material region V. Expression (4.1), in which we identify the virtual power of

internal and contact forces, respectively, is in the classical form of the principle of virtual power for a finite body in quasi-statics and the absence of body loads (cf. Maugin 1980). Following this last reference we let the reader check by himself that the translational and rotational parts of a virtual rigid-body motion render the global form of equations (2.21) and (2.27) extended to V; that is,

$$\int_{\partial V} \mathcal{T}\,\mathrm{d}S = 0, \quad \int_{\partial V} (\boldsymbol{N}\cdot\mathcal{M} + \boldsymbol{x}\times\mathcal{T})\,\mathrm{d}S = \boldsymbol{0}.$$

In the same regularity conditions, through a material-space integration procedure, the canonical equations (3.4) and (3.6) yield

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V W\,\mathrm{d}V = \int_{\partial V} (\mathcal{T}\cdot\boldsymbol{v} + M\cdot\dot{\mathfrak{X}})\,\mathrm{d}S \tag{4.3}$$

and

$$\int_V \boldsymbol{f}^{\mathrm{inh}}\,\mathrm{d}V + \int_{\partial V} \boldsymbol{N}\cdot\boldsymbol{b}\,\mathrm{d}S = \boldsymbol{0}, \tag{4.4}$$

where the latter is understood component wise. For a materially homogeneous region V, (4.4) further reduces to the simple strict conservation law

$$\int_{\partial V} \boldsymbol{N}\cdot\boldsymbol{b}\,\mathrm{d}S = \boldsymbol{0}, \tag{4.5}$$

i.e.

$$\int_V (W\boldsymbol{N} - \mathcal{T}\cdot\boldsymbol{F} - M\cdot\nabla_{\mathrm{R}}\mathfrak{X})\,\mathrm{d}S = \boldsymbol{0}. \tag{4.6}$$

Here equations (4.3)–(4.5) serve no special purpose except perhaps as checks of a numerical scheme—this is classical for the energy, obviously original for equations (4.4) and (4.5). This type of equation becomes of real interest if the fields are not sufficiently regular. This is the case in the paradigmatic problem of fracture such as met in the possible progress of a straight-through crack that we briefly examine now. Indeed, for instance, in a homogeneous body, the local equation $\div_{\mathrm{R}}\boldsymbol{b} = \boldsymbol{0}$ is *not integrable* at the tip A of a crack C in classical elasticity; this is due to the singularity order of $\boldsymbol{b}$, which being at least quadratic in the fields, has a singularity which, in two dimensions, overcomes the bulk integration. Thus, around a singular point such as A,

$$\int_{V_A} (\div_{\mathrm{R}}\boldsymbol{b})\,\mathrm{d}V \neq \boldsymbol{0} \tag{4.7}$$

as V_A shrinks to zero. The same holds true for $\partial W/\partial t$. As a consequence, the lack of integrability will manifest itself in the global equation (4.7) as a global material force $\mathcal{F}$ similar in its effects to the global force of inhomogeneity present in (4.4)—a material covector—called the driving force on the tip of the crack. That is, this 'force', fictitious in a way since not acting on matter per unit mass but acting on a field singularity, is defined by

$$\mathcal{F} := \lim_{V_A\to 0}\int_{\partial V_A} \boldsymbol{N}\cdot\boldsymbol{b}\,\mathrm{d}S_A. \tag{4.8}$$

But we need not apply the limit procedure if $\mathcal{F}$ is along the direction of the crack and the faces of the crack $\mathcal{C}$ are free (homogeneous boundary conditions along the

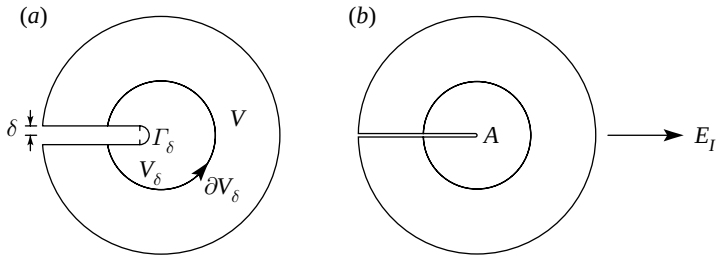


Figure 1. (*a*) Notch, (*b*) limit crack.

faces of $\mathcal{C}$ (cf. the integrand in equation (4.6))), because then $\mathcal{F}\cdot\boldsymbol{N}=0$ along the faces of the crack and the integral in (4.8) is obviously contour independent.

Let us apply this to the (two-dimensional) crack problem in figure 1*b*, where the straight-through crack is considered as the limit of a family of notches (figure 1*a*) indexed by the half thickness δ (also the radius of the front of progress Γ_δ). ∂V_δ is the path of integration in the material for a given δ and front Γ_δ. We have thus

$$\mathcal{F}(A)=\lim_{\delta\to 0}\int_{\partial V_\delta}\boldsymbol{N}\cdot\boldsymbol{b}\,\mathrm{d}S. \tag{4.9}$$

It is immediately checked that this can also be written as

$$\mathcal{F}(A)=\lim_{\delta\to 0}\int_{\Gamma_\delta}\boldsymbol{N}_\Gamma\cdot\boldsymbol{b}\,\mathrm{d}S=\lim_{\delta\to 0}\int_{\Gamma_\delta}W\boldsymbol{N}_\Gamma\,\mathrm{d}S, \tag{4.10}$$

in which we recognize the outflux ($\boldsymbol{N}_\Gamma$ is oriented toward the material; the fronts Γ_δ have zero surface load) of Eshelby stress through the tip of the crack. This follows from the fact that (4.5) holds for the regular domain V_δ encircled by ∂V_Γ, Γ_δ (free of loads; hence the reduction of $\boldsymbol{N}\cdot\boldsymbol{b}$ to $W\boldsymbol{N}$) and the two (free) faces of the crack. Now we can perform a similar reasoning on the energy equation. To that purpose we first write the global energy equation for the regular domain V_δ, δ fixed, with a notch front Γ_δ that progresses at material velocity $\bar{\boldsymbol{V}}$ in the $\boldsymbol{E}_\mathrm{r}$ direction. Accounting for the motion of that part of the surface and using the Reynolds transport theorem for an evolutive domain of integration, the volume integral of (2.21) yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\delta}W\,\mathrm{d}V+\mathcal{G}=\int_{\partial V_\delta}(\mathcal{T}\cdot\boldsymbol{v}+M\cdot\dot{\mathfrak{X}})\,\mathrm{d}S, \tag{4.11}$$

where the $\mathcal{G}$ is defined by

$$\mathcal{G}(\Gamma_\delta)=\int_{\Gamma_\delta}W(\bar{\boldsymbol{V}}\cdot\boldsymbol{N}_\Gamma)\,\mathrm{d}S. \tag{4.12}$$

Equivalently, this is also given by

$$\mathcal{G}(\Gamma_\delta)=\int_{\partial V_\delta}\{W(\bar{\boldsymbol{V}}\cdot\boldsymbol{N})+\mathcal{T}\cdot\boldsymbol{v}+M\cdot\dot{\mathfrak{X}}\}\,\mathrm{d}S-\frac{\mathrm{d}}{\mathrm{d}t}\int_{V_\delta}W\,\mathrm{d}V. \tag{4.13}$$

Now we must proceed to the 'sharp crack' limit as δ goes to zero. Following Dascalu & Maugin (1993), we note that P denoting any point of Γ_δ and writing the general motion as $\chi(\boldsymbol{X},t)=\chi(\boldsymbol{X}_P+r,t)$, so that, by differentiation,

$$\boldsymbol{v}=-\boldsymbol{F}\cdot\bar{\boldsymbol{V}}+\frac{\partial\bar{\chi}}{\partial t}, \tag{4.14\,a}$$

$$\bar{\boldsymbol{V}}_P = \frac{\partial \boldsymbol{X}_P}{\partial t}, \tag{4.14 b}$$

where $\chi(\boldsymbol{X}, t) = \chi(\boldsymbol{X}, \boldsymbol{t})$; by convection we have

$$\boldsymbol{V} = \bar{\boldsymbol{V}} - \boldsymbol{F}^{-1} \cdot \frac{\partial \bar{\chi}}{\partial t}, \quad \bar{\boldsymbol{V}} = \bar{V}_{\mathrm{r}} \boldsymbol{E}_{\mathrm{r}}. \tag{4.15}$$

Let all points P of Γ_δ be in uniform motion $\bar{\boldsymbol{V}} = \bar{\boldsymbol{V}}(t)$, then the deformation may be assumed the same at all points P and thus $\partial \bar{\chi}/\partial t = \boldsymbol{0}$. Similarly, we shall obtain, for $\dot{\mathfrak{X}}$ (compare to (4.14 a)),

$$\dot{\mathfrak{X}} = -\bar{\boldsymbol{V}} \cdot \nabla_{\mathrm{R}} \mathfrak{X} + \text{regular term}. \tag{4.16}$$

The above two expressions (4.15) and (4.16) can also be interpreted in the limit at δ goes to zero, by saying that the time derivatives of the basic fields behave regularly in the frame moving with the tip A, i.e. the derivatives defined by

$$\frac{\partial}{\partial t} \chi = \boldsymbol{v} + \bar{\boldsymbol{V}} \cdot \nabla_{\mathrm{R}} \chi, \quad \frac{\partial}{\partial t} \mathfrak{X} = \dot{\mathfrak{X}} + \bar{\boldsymbol{V}} \cdot \nabla_{\mathrm{R}} \mathfrak{X} \tag{4.17}$$

are regular at A, so that we have the essential behaviour indicated in (4.16), an analog formula holding for χ. This is a result of Gurtin (1979) here generalized to the case of polar elastic media. To that purpose we must remark that, according to the work of several authors, the velocity field of the classical motion and the micromotion, equivalently the usual strain and microstrain, have the same singularity order at the tip of a sharp crack as in classical elasticity (cf. Maugin (1992) for classical elasticity, Sternberg & Muki (1967), Ejike (1969) and Atkinson & Leppington (1974) for coupled-stress and micropolar elasticity—it is this mathematical fact which softened the enthusiasm for the couple-stress theory in the 1970s).

As only the component $\bar{V}_{\mathrm{I}}$ of $\bar{\boldsymbol{V}}$ is non-zero, scalar multiplication of (4.10) by $\bar{\boldsymbol{V}}$, on account of the above stated relations, yields the following remarkable result:

$$\mathcal{G}(A) = \lim_{\delta \to 0} \mathcal{G}(\delta) = \bar{\boldsymbol{V}} \cdot \mathcal{F}(A). \tag{4.18}$$

This is a global result akin to the local one (3.9) because it relates a quantity $\mathcal{G}(A)$, known as the energy-release rate, to the power expended by the driving force in a material motion of velocity $\bar{\boldsymbol{V}}$ (here of the crack tip). But the quantity (where V evolves with the crack length)

$$\Phi(V) \equiv \int_{\partial V} (\mathcal{T} \cdot \boldsymbol{v} + M \cdot \dot{\mathfrak{X}}) \, \mathrm{d}V - \frac{\mathrm{d}}{\mathrm{d}t} \int_V W \, \mathrm{d}S, \tag{4.19}$$

the opposite of the total potential energy rate, is also the time rate of growth of total entropy, i.e. dissipation rate in the irreversible quasi-static evolution of the domain V by cracking (cf. Maugin 1992, ch. 7). Thus expression (4.18) in fact corresponds to dissipation, a quantity which cannot be negative. Hence the dissipation inequality

$$\mathcal{G}(A) = \bar{\boldsymbol{V}} \cdot \mathcal{F}(A) \geqslant 0; \tag{4.20}$$

this supports a formulation of a functional relationship between $\bar{\boldsymbol{V}}$ and $\mathcal{F}$ in the framework of irreversible thermodynamics or one of its 'nonlinear' generalizations. Realistically, this relation will be of the 'threshold type' (just like in plasticity), the actual value of $\mathcal{F}$ being computed from an evaluation of the field solution at each given length of the crack during extension and this value being compared at each

increment to a characteristic material value proper to each material (the fracture toughness or tenacity), this comparison deciding of a further evolution (growth) or no evolution.

5. Accounting for inertia

If inertia is accounted for, then we must consider the true Lagrangian density per unit volume of $\mathcal{K}_{\mathrm{R}}$

$$L = K - W, \tag{5.1}$$

where K, the kinetic energy per unit volume of $\mathcal{K}$, must involve the velocity fields $\{\boldsymbol{v}, \dot{\mathfrak{X}}\}$ or $\{\boldsymbol{v}, \nu\}$ in one way or another. For a classical type of inertia, e.g. no gyroscopic term, K is taken a homogeneous function of degree two of these velocities. That is,

$$2K \equiv \frac{\partial K}{\partial \boldsymbol{v}} \cdot \boldsymbol{v} + \frac{\partial K}{\partial \dot{\mathfrak{X}}^{i}_{\cdot K}} \dot{\mathfrak{X}}^{i}_{\cdot K}. \tag{5.2}$$

In the case of media without microstructure we obviously have $\mathcal{K} = \frac{1}{2}\rho_0(\boldsymbol{X})\boldsymbol{v}^2$, where $\rho_0(\boldsymbol{X})$ is the matter density at $\mathcal{K}_{\mathrm{R}}$, possibly a function of the material point $\boldsymbol{X}$ if the material presents inertial inhomogeneities, so that we have the following material conservation law:

$$\left.\frac{\partial \rho_0}{\partial t}\right|_{\boldsymbol{X}} = 0. \tag{5.3}$$

The general solution $\rho_0(\boldsymbol{X})$ of this, and the actual density $\rho(\boldsymbol{x}, t)$ at placement $\boldsymbol{x}$ in $\mathbb{E}^3$, are related by the integral form of the continuity equation

$$\rho_0 = \rho J. \tag{5.4}$$

The question now is how do we generalize this to the present case with a rotational microstructure? The simplest solution of the functional equation (5.2) reads

$$K = \tfrac{1}{2}\rho_0(\boldsymbol{X})(\boldsymbol{v}^2(\boldsymbol{X}, t) + L^{\cdot\cdot KL}_{ij}(\boldsymbol{X})\dot{\mathfrak{X}}^{i}_{\cdot K}\dot{\mathfrak{X}}^{j}_{\cdot L}), \tag{5.5}$$

where the geometrical object L is symmetric in indices i and j and also in indices K and L. On account of (2.13) and (2.14), this can also be written as the apparently simpler form

$$K = \tfrac{1}{2}\rho_0(\boldsymbol{X})(\boldsymbol{v}^2(\boldsymbol{X}, t) + \sigma(\boldsymbol{X}, t) \cdot \nu(\boldsymbol{X}, t)), \tag{5.6}$$

where the (axial and spatial) spin vector σ and the inertia tensor of the microstructure j (a symmetric spatial tensor with six independent components at most) are defined by

$$\sigma = j \cdot \nu \tag{5.7}$$

and (in components)

$$j^{pq} = \varepsilon^{p}_{\cdot ik}\varepsilon^{q}_{\cdot jl}L^{ijKL}(\boldsymbol{X})\mathfrak{X}^{k}_{\cdot K}\mathfrak{X}^{l}_{\cdot L}. \tag{5.8}$$

It was shown by the Eringen school in the 1960s that if the material inertial tensor $J^{PQ} = j^{pq}(\mathfrak{X}^{-1})^{P}_{\cdot p}(\mathfrak{X}^{-1})^{Q}_{\cdot q}$ satisfies the material conservation law (cf. equation (5.3)),

$$\left.\frac{\partial J^{PQ}}{\partial t}\right|_{\boldsymbol{X}} = 0, \tag{5.9}$$

then, in turn, j satisfies the following 'spatial' balance law (parentheses around a set of indices indicate symmetrization):

$$\left(\frac{\mathrm{d}j^{kl}}{\mathrm{d}t}\right) - 2\bar{\nu}^{(k}_{\cdot m}\bar{\nu}^{l)m} = 0. \tag{5.10}$$

The generalization of equations (2.21), (2.22), (2.27) and (3.4) to the case of full dynamics is easily formulated by adding terms (the third and fourth after computation)

$$-\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \boldsymbol{v}}\right), \quad -\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\mathfrak{X}}}\right), \quad -\frac{\partial}{\partial t}\sigma\Big|_{\boldsymbol{X}}, \quad -\frac{\partial}{\partial t}L\Big|_{\boldsymbol{X}},$$

repectively, in the right-hand side of these equations. Thus the replacement equations corresponding to the balance laws (2.21), (2.27) and (3.3) read

$$\frac{\partial}{\partial t}\rho_0\boldsymbol{v}\Big|_{\boldsymbol{X}} - \div_{\mathrm{R}}\boldsymbol{T} = \boldsymbol{0}, \tag{5.11}$$

$$\frac{\partial}{\partial t}\rho_0\sigma\Big|_{\boldsymbol{X}} - \div_{\mathrm{R}}\boldsymbol{M} - \boldsymbol{F} \mathbin{\dot{\times}} \boldsymbol{T} = \boldsymbol{0}, \tag{5.12}$$

and

$$\frac{\partial}{\partial t}(K+W)\Big|_{\boldsymbol{X}} - \div_{\mathrm{R}}(\boldsymbol{T}\cdot\boldsymbol{v} + \boldsymbol{M}\cdot\nu) = 0, \tag{5.13}$$

which are, in the classical 'material' form, given in Eringen & Kafadar (1976) in the absence of dissipative processes and external 'forces'. Much more intriguing is the generalization of the 'pseudo-momentum' equation—here a completely new equation—because the fully dynamical version of that canonical equation will capture all momenta, whether linear or angular, in the notion of pseudo-momentum. Indeed, the canonical definition of material momentum with a generalized motion (2.16) labelled in an ordered sequence of independent components (generalized coordinates) by $q^\alpha(\boldsymbol{X},t)$, $\alpha = 1,\ldots,9$, is (cf. Maugin 1993; Soper 1976) that of a material covector $\mathcal{P}$ such that

$$\mathcal{P} := -\sum_\alpha (\nabla_{\mathrm{R}} q^\alpha)\left(\frac{\partial L}{\partial \dot{q}^\alpha}\right). \tag{5.14}$$

Accordingly, on account of (5.1) and (5.5), we directly obtain

$$\mathcal{P} = -\rho_0(\boldsymbol{F}^{\mathrm{T}}\cdot\boldsymbol{v} - \mathcal{L}\cdot\nu), \tag{5.15}$$

where the linear operator $\mathcal{L}$, a two-point tensor field, is defined in components by

$$\mathcal{L}^{\cdot n}_{L} := -\mathfrak{X}^i_{\cdot K,L}\varepsilon_{jmn}L^{\cdot\cdot KQ}_{ij}\mathfrak{X}^m_{\cdot Q}. \tag{5.16}$$

On account of (2.4) and the second equation of (2.6), we check that (5.15) can also be written as

$$\mathcal{P} = \rho_0(\boldsymbol{C}\cdot\boldsymbol{V} + \mathcal{L}\cdot\nu), \tag{5.17}$$

which clearly exhibits the contribution of the classical motion *and* the micromotion of rotation velocity ν to pseudo-momentum. The corresponding balance equation is the following generalization of equation (3.6):

$$\frac{\partial \mathcal{P}}{\partial t}\Big|_{\boldsymbol{X}} - \div_{\mathrm{R}}\boldsymbol{b} = \boldsymbol{f}^{\mathrm{inh}}, \tag{5.18}$$

where, now,

$$\boldsymbol{b} = -(L\mathbf{1}_{\mathrm{R}} + \boldsymbol{T}\cdot\boldsymbol{F} + \mathfrak{M} : \nabla_{\mathrm{R}}\mathfrak{X}), \quad \boldsymbol{f}^{\mathrm{inh}} = \left(\frac{\partial L}{\partial \boldsymbol{X}}\right)_{\mathrm{expl}}. \tag{5.19}$$

Expression (5.15) with its two contributions is not without recalling what was obtained for liquid crystals—another continuum system exibiting two vectorial degrees of freedom endowed with inertia—in Maugin & Trimarco (1995*a*). However, there is more to it as we know that the Hamiltonian-mechanics definition of $\mathcal{P}$ is (Maugin 1993)

$$\mathcal{P} = \frac{\partial L}{\partial \boldsymbol{V}}, \tag{5.20}$$

because of the first equation of (2.6) and the fact that $\mathcal{P}$ is the generalized momentum associated with the generalized coordinates $\boldsymbol{X}$. But the latter definition (5.20) truly holds when all fields are expressed in terms of the Eulerian independent variables $(\boldsymbol{x}, t)$; what we called the inverse-motion description. In these variables, the Lagrangian density $\hat{L}$ per unit volume of the *actual* configuration $\mathcal{K}_t$ will obviously read

$$\hat{L} = \tfrac{1}{2}\rho(\boldsymbol{V}\cdot\boldsymbol{C}\cdot\boldsymbol{V} + L_{ij}^{\cdot\cdot KL}\dot{\mathfrak{X}}^{i}_{\cdot K}\dot{\mathfrak{X}}^{j}_{\cdot L}) - J\hat{W}, \tag{5.21}$$

because $\boldsymbol{v}^2 \equiv \boldsymbol{V}\cdot\boldsymbol{C}\cdot\boldsymbol{V}$, but with

$$\dot{\mathfrak{X}}^{i}_{\cdot K} := \left(\frac{\partial \mathfrak{X}^{i}_{\cdot K}}{\partial t}\right)\Bigg|_{\boldsymbol{X}} = \left(\frac{\partial \mathfrak{X}^{i}_{\cdot K}(\boldsymbol{x}, t)}{\partial t}\right)\Bigg|_{\boldsymbol{x}} - \boldsymbol{V}\cdot\nabla_{\mathrm{R}}\mathfrak{X}^{i}_{\cdot K}. \tag{5.22}$$

Then we immediately check that the definition (5.20) applied to (5.21) yields directly (5.17), but with ρ replacing ρ_0. ■

Equations (5.3), (5.9), (5.11), (5.12), (5.13) and (5.18) are the complete set of balance laws in the dynamical, albeit non-dissipative, case. We let the reader study the consequences of this fully dynamic formulation on the formulae given in § 4 for fracture (by following the work of Dascalu & Maugin (1993) for classical elasticity).

Remarks on time and length scales

Let T and L be macroscopic time and length scales associated with the body and physical data, e.g. frequency or transient time of externally applied forces, and the typical size of the specimen. Taking account of inertia introduces two characteristic times t_{M} and t_{MM} related, respectively, to the inertia of classical motion χ (deformation) and micromotion $\mathfrak{X}$. Characteristic velocities c_{M} and c_{MM}, frequencies ω_{M} and ω_{MM} and wavelengths λ_{M} and λ_{MM}, are thus introduced which must be compared to one another and to those which can be associated with external stimuli. Depending on the case studied and the microstructure included, one may discard (i) the inertia of micromotion compared to that of the classical motion (a situation that we may refer to as 'quasi-microstatics'; generally speaking, micromotion gives rise to high-frequency phenomena which are in the 'optical' range (cf. Pouget *et al.* (1986*a*,*b*) for a physical example with numerical figures)), or (ii) the inertial time scale of the micromotion compared to the time scale of the viscous process associated with that micromotion (this is done in liquid crystals where rotational inertia is such that its effects are most often neglected compared to those of rotational viscosity, whose own time scale is compared to that of the macromotion, e.g. via the Deborah number in rheology (see de Gennes 1974)).

The introduction of the characteristic wavelength $\lambda_{\rm MM}$ is more problematic as, if $\lambda_{\rm M}$ is naturally to be compared to L (in many cases $\lambda_{\rm M} \ll L$), $\lambda_{\rm MM}$—which naturally satisfies $\lambda_{\rm MM} \ll \lambda_{\rm M}$—is of the order of the characteristic length $\ell_{\rm MM}$ related to the introduction of the micromotion gradients $\nabla_{\rm R}\mathfrak{X}$—and to the spatial range of interactions in a lattice description (e.g. first-neighbour interactions for rotational effects in a crystal equipped with a polar group such as $NaNO_2$)—and/or of the order of the characteristic length $\ell_{\rm R}$ related to *material* inhomogeneities. We have the following sensible orders of magnitude:

$$\ell_{\rm MM} \cong \frac{|\mathfrak{X}|}{|\nabla_{\rm R}\mathfrak{X}|} \cong \frac{\|\boldsymbol{C}\|}{\|\mathfrak{G}\|}, \quad \ell_{\rm R} \cong \frac{|\bar{\mathcal{L}}|}{\|\boldsymbol{f}^{\rm inh}\|}. \tag{5.23}$$

Whenever $O(\ell_{\rm MM}) = O(\ell_{\rm R})$ (the case of micro-inhomogeneities), both microstructure and material inhomogeneities are to be included in the treatment. This is the case if we have microcracks, which are micro-inhomogeneities in their own right, as their presence would manifest through material forces of quasi-inhomogeneities (compare to § 4). This is also what occurs in the problem of phase-transition fronts—see § 7 where this is taken of mathematically zero thickness—where the true transition zone (the material inhomogeneity) is of the order of a few lattice spaces, i.e. of the order of $\ell_{\rm MM}$. Similarly, for an external excitation (high frequency) of wavelength $\lambda_{\rm e} \cong \ell_{\rm MM}$, the microstructure is excited; it must necessarily be taken into account. The same argument holds for material inhomogeneities. This corresponds to the reinterpretation of the continuum in its discrete, lattice description. Obviously, the same questions must be raised in the discretization of the continuum in numerical schemes . The capture of scale effects obviously requires a mesh whose characteristc size be much smaller than $\ell_{\rm MM}$ or $\ell_{\rm R}$, as otherwise, parodying G. Bachelard (1927), 'the mesh may be too coarse, stepping over the discontinuities of the punctiform distribution, and missing the 'confusion' that would prevail'; but this 'confusion' indeed is the rich phenomenon, the complex behaviour , on which micromechanics focus attention. In summary, scale effects related to the presence of both $\nabla_{\rm R}\mathfrak{X}$ and $\boldsymbol{X}$ in the functional dependence (2.17) will play a role in many instances both physical (e.g. in the presence of microcracks, micro-inclusions, short-wavelength excitation, existence of field singularities of the point, line and surface types) and numerical. Because of this possible matching of scales, the recent progress in numerical implementation allowing for fine meshes and fast computations thus increases the interest in the model of polar media. Note that the basic invariance of physics mentioned in (2.18 *a*) has nothing to do with the scale effects which are here of purely material origin, i.e. are related to the $\boldsymbol{X}$ dependence of both physical properties and field solution.

6. Thermoelastic conductors

Although heat conduction will be the only dissipative process present, we have no longer the possibility to base the reasoning on a variational principle and the exploitation of Noether's principle. In this case we are limited to the 'naive', but efficient, direct method already used in § 3. That is, we assume that the physical balance laws have already been established by other means (postulate of global balance laws) and the general constitutive equations are obtained via the now standard thermomechanical approach (cf. Kafadar & Eringen 1976) for the case of thermoelastic polar media in finite strains. The essentially new field variable—compared to

previous sections—is the absolute temperature $\theta > 0$ and we must then distinguish between internal and free (Helmholtz) energies. We call these, respectively, E and W per unit volume at $\mathcal{K}_{\mathrm{R}}$. Then the basic local balance laws, all regular material points $\boldsymbol{X}$ in the absence of body force and couple, and body heat source, are those of mass, micro-inertia, physical momentum, moment of momentum, energy (first law of thermodynamics) and entropy in the following 'material' form:

$$\frac{\partial}{\partial t}\rho_0\bigg|_{\boldsymbol{X}} = 0, \tag{6.1}$$

$$\frac{\partial}{\partial t}J^{KL}\bigg|_{\boldsymbol{X}} = 0, \tag{6.2}$$

$$\frac{\partial}{\partial t}\boldsymbol{p}\bigg|_{\boldsymbol{X}} - \div_{\mathrm{R}}\boldsymbol{T} = \boldsymbol{0}, \tag{6.3}$$

$$\frac{\partial}{\partial t}\rho_0\sigma\bigg|_{\boldsymbol{X}} - \div_{\mathrm{R}}\boldsymbol{M} - \boldsymbol{F} \mathbin{\dot{\times}} \boldsymbol{T} = \boldsymbol{0}, \tag{6.4}$$

$$\frac{\partial}{\partial t}(K+E)\bigg|_{\boldsymbol{X}} - \nabla_{\mathrm{R}}\cdot(\boldsymbol{T}\cdot\boldsymbol{v} + \boldsymbol{M}\cdot\nu - \boldsymbol{Q}) = \boldsymbol{0}, \tag{6.5}$$

$$\theta\frac{\partial}{\partial t}S\bigg|_{\boldsymbol{X}} + \nabla_{\mathrm{R}}\cdot\boldsymbol{Q} = 0, \tag{6.6}$$

where $\boldsymbol{Q}$ is the material heat (out)flux and S is the entropy per unit volume at $\mathcal{K}_{\mathrm{R}}$. Equation (6.6) can also be written as

$$\frac{\partial}{\partial t}S\bigg|_{\boldsymbol{X}} + \nabla_{\mathrm{R}}\cdot\left(\frac{\boldsymbol{Q}}{\theta}\right) = \frac{d}{\theta}, \tag{6.7}$$

where

$$d = -\boldsymbol{Q}\cdot\nabla_{\mathrm{R}}(\ln\theta). \tag{6.8}$$

The second law of thermodynamics imposes that $d \geqslant 0$, while the constitutive equations for non-dissipative processes have been deduced, by use of the 'thermodynamical admissibility' argument, as

$$\boldsymbol{T} = \left(\frac{\partial\bar{W}}{\partial\boldsymbol{F}}\right)^{\mathrm{T}}, \quad \mathfrak{M} = \left(\frac{\partial\bar{W}}{\partial(\nabla_{\mathrm{R}}\mathfrak{X})}\right)^{\mathrm{T}}, \quad S = -\frac{\partial\bar{W}}{\partial\theta}, \quad \boldsymbol{M} \equiv \mathfrak{M}\dot{\times}\mathfrak{X}, \tag{6.9}$$

with

$$W = \bar{W}(\boldsymbol{F}, \mathfrak{X}, \nabla_{\mathrm{R}}\mathfrak{X}, \theta; \boldsymbol{X}) = E - S\theta. \tag{6.10}$$

In so far as $\boldsymbol{Q}$ is concerned, it is only subjected to the non-negativeness condition imposed on d and to the continuity condition

$$\boldsymbol{Q}(\boldsymbol{F}, \mathfrak{X}, \nabla_{\mathrm{R}}\mathfrak{X}, \theta, \nabla_{\mathrm{R}}\theta; \boldsymbol{X}) \to \boldsymbol{0} \quad \text{as } \nabla_{\mathrm{R}}\theta \to \boldsymbol{0}. \tag{6.11}$$

The set of equations (6.1) through (6.11) leaves us with very little possibility of manoeuvre. In order to obtain the local balance of pseudo-momentum, we must multiply (6.3) to the left by $\boldsymbol{F}^{\mathrm{T}}$ and, by integration by parts while taking account of (6.10), (6.9) and the expressions of σ and $\mathcal{L}$, show that the resulting covectorial

material equation reads

$$\frac{\partial}{\partial t}\mathcal{P}\Big|_{\boldsymbol{X}} - \div_{\mathrm{R}}\boldsymbol{b} = \boldsymbol{f}^{\mathrm{inh}} + \boldsymbol{f}^{\mathrm{th}}, \tag{6.12}$$

where $\mathcal{P}$ has already been defined and

$$\boldsymbol{b} = -(L^{\mathrm{th}}\mathbf{1}_{\mathrm{R}} + \boldsymbol{T}\cdot\boldsymbol{F} + \mathfrak{M} : \nabla_{\mathrm{R}}\mathfrak{X}), \tag{6.13}$$

$$L^{\mathrm{th}} := K - \bar{W}(.,.,.,\theta;\boldsymbol{X}), \tag{6.14}$$

$$\boldsymbol{f}^{\mathrm{inh}} := \left(\frac{\partial L^{\mathrm{th}}}{\partial \boldsymbol{X}}\right)_{\mathrm{expl}}, \tag{6.15}$$

$$\boldsymbol{f}^{\mathrm{th}} := S\nabla_{\mathrm{R}}\theta, \tag{6.16}$$

in which L^{th} is some kind of effective Lagrangian built from the free energy; note the difference with (6.5), where it is the internal energy (a function of S) that is involved. We see that a local non-uniformity in temperature plays the same role as a material inhomogeneity—via $\boldsymbol{f}^{\mathrm{th}}$—in the balance of pseudo-momentum. This quasi-inhomogeneity force remains there even when the material itself is homogeneous. Equation (6.12) is the polar-elasticity generalization of the result of Epstein & Maugin (1995). This formula would help one formulate the problem of fracture in the dynamical thermoelasticity of polar materials by generalizing the recent work of Dascalu & Maugin (1995). However, much more interesting here is the fact that equations (6.1)–(6.12) form the necessary background for the study of the progress of phase-transition fronts in polar elastic crystals as, obviously, temperature effects are most relevant then since the transition, if dissipative, must materialize in a localized heat source.

7. Coherent phase-transition fronts

In a classical thermoelastic crystal, a coherent phase-transition front is defined as a homothermal discontinuity surface Σ separating two phases and across which, not only temperature is continuous (since the transition occurs at a temperature where the two phases coexist), but also the material velocity $\boldsymbol{V}$. This condition, expressed here in terms of a velocity, simply means that the lattice sites at the front belong to the two phases; we call this the continuity of lattice sites. In other words, although, in general, two different crystalline structures exist on both sides of Σ, there are no dislocations at Σ (cf. Maugin & Trimarco 1995*a*–*c*). This is a rather rigid definition which we easily conceive as being seldom verified in practice. The conditions imposed in the case of polar crystals are even more severe since the microrotations of material points belonging to the two crystalline systems at Σ must obviously be synchronized in time. Otherwise, we would have some kind of disclinations present at Σ. We thus consider the very theoretical and pure case where, indeed, we can satisfy these stringent continuity conditions. Those relating to placement and microrotations are in fact applicable at the actual placement $\boldsymbol{x}$. This is why it is the generalized velocities based on $\boldsymbol{X}(\boldsymbol{x},t)$ and $\mathfrak{X}(\boldsymbol{x},t)$—and not $\mathfrak{X}(\boldsymbol{X},t)$—which are to be continuous at Σ. In all, therefore, we should apply the following continuity conditions:

$$[\theta] = 0, \tag{7.1 a}$$

$$[\boldsymbol{V}] = \mathbf{0}, \tag{7.1 b}$$

$$\left[\left(\frac{\partial \mathfrak{X}}{\partial t}\right)_{\boldsymbol{x}}\right] = \mathbf{0}, \tag{7.1 c}$$

where the symbolism $[\cdot]$ denotes, as usual, the jump of a quantity between its value right ahead of the front Σ (side 'plus') and right behind Σ (side 'minus'), the unit normal $\boldsymbol{N}$ to Σ being oriented from Ω^- to Ω^+ if these are the two parts of the material body Ω occupied by phase I (or 'minus') and II (or 'plus'), respectively (figure 2). A direct consequence of (7.1 *c*) and (7.1 *b*) is that

$$\left[\left(\frac{\partial \mathfrak{X}}{\partial t}\right)_{\boldsymbol{X}}\right] = -\bar{\boldsymbol{V}} \cdot [\nabla_{\mathrm{R}} \mathfrak{X}], \tag{7.2}$$

where $\bar{\boldsymbol{V}}$ is the material vectorial velocity of Σ. We denote by $\bar{\theta}$ the temperature field at Σ. The relevant question now is the following. If equations (6.1)–(6.12) are satisfied in the two adjacent regions $\Omega^{\pm}$, for energies $\bar{W}^{\pm}$ corresponding to different symmetries, and thus different material coefficients, respectively, but each phase being materially homogeneous ($\boldsymbol{f}^{\mathrm{inh}} \equiv \mathbf{0}$ in $\Omega^{\pm}$), then what are the jump relations satisfied by various fields at Σ? In particular, is there one such relation which, in fact, *governs* the transition? To the first part of the question we answer in the following pragmatic way. The jump relations associated with true conservation laws such as equations (6.1)–(6.3) and (6.5) are known from the theory of hyperbolic systems and the formalism of weak solutions to be obtained formally by replacing operators $\partial/\partial t|_{\boldsymbol{X}}$ and ∇_{R} by $-\bar{V}_N[\cdot]$ and $\boldsymbol{N} \cdot [\cdot]$, respectively, where $\bar{V}_N := \bar{\boldsymbol{V}} \cdot \boldsymbol{N}$. These equations, together with the spin jump equation are recalled by Eringen & Kafadar (1976) for polar media. Up to the notation of these authors which differs from ours (in particular, material heat flux is of opposite sign in the two notations), these read

$$\bar{V}_N[\rho_0] = 0, \tag{7.3}$$

$$\bar{V}_N[J^{KL}] = 0, \tag{7.4}$$

$$\boldsymbol{N} \cdot [\boldsymbol{T} + \bar{\boldsymbol{V}} \otimes \boldsymbol{p}] = \mathbf{0}, \tag{7.5}$$

$$\boldsymbol{N} \cdot [\boldsymbol{M} + \bar{\boldsymbol{V}} \otimes \rho_0 \sigma] = \mathbf{0}, \tag{7.6}$$

$$\boldsymbol{N} \cdot [(K+E)\bar{\boldsymbol{V}} + (\boldsymbol{T} \cdot \boldsymbol{v} + \mathfrak{M} \cdot \dot{\mathfrak{X}} - \boldsymbol{Q})] = 0. \tag{7.7}$$

For a truly propagating front, $\bar{V}_N \neq 0$, equations (7.3) and (7.4) imply the continuity of both ρ_0 and J^{KL}:

$$[\rho_0] = 0, \quad [J^{KL}] = 0. \tag{7.8}$$

As to equations (6.6), (6.7) and (6.12), which are not in conservative form, we can only surmise that, in addition to the usual jump terms, we will have surface sources q_Σ, d_Σ and $\boldsymbol{f}_\Sigma$, such that we can *a priori* write the jump relations

$$\boldsymbol{N} \cdot [\theta S \bar{\boldsymbol{V}} - \boldsymbol{Q}] - q_\Sigma = 0, \tag{7.9}$$

$$\boldsymbol{N} \cdot [S\bar{\boldsymbol{V}} - (\boldsymbol{Q}/\theta)] = d_\Sigma \geqslant 0, \tag{7.10}$$

and

$$\boldsymbol{N} \cdot [\boldsymbol{b} + \bar{\boldsymbol{V}} \otimes \mathcal{P}] + \boldsymbol{f}_\Sigma = 0. \tag{7.11}$$

Here q_Σ, d_Σ and $\boldsymbol{f}_\Sigma$ are, respectively, a surface heat source, a surface dissipation rate and a surface quasi-inhomogeneity force. That a 'force' such as the latter intervenes in the theory is not surprising as the translation of Σ in $\boldsymbol{X}$ space places the difference in

phase on the two sides of Σ, i.e. a material inhomogeneity, *per se*, in evidence. By the duality inherent in mechanics, a material 'inhomogeneity force', or a force acting in a similar way (hence the qualification of 'quasi-homogeneity') ought to be generated. It is clear, however, that the new surface quantities introduced in equations (7.9) through (7.11) are not all independent. First, on account of (7.1), from (7.9) and (7.10) we obviously have

$$d_\Sigma = q_\Sigma/\bar{\theta}, \tag{7.12}$$

where $\bar{\theta}$ is the temperature shared by points of $\Omega^\pm$ at Σ. The proof of the next statement requires more work for which we give the detail only for the case of quasi-statics (inertia neglected). Following along the examples provided by equations (3.9) and (4.20), one tries to evaluate the power expended by the material force $\boldsymbol{f}_\Sigma$ in the velocity field $\bar{\boldsymbol{V}}$ of points of Σ where it acts. In the simplified frame now considered, we have

$$P(\boldsymbol{f}_\Sigma) = \boldsymbol{f}_\Sigma \cdot \bar{\boldsymbol{V}} = -\boldsymbol{N} \cdot [\boldsymbol{b}] \cdot \bar{\boldsymbol{V}}, \tag{7.13}$$

where $\boldsymbol{b}$ is given by expression (6.13) with K neglected. Simultaneously, evaluating $\boldsymbol{N} \cdot [\boldsymbol{Q}]$ from (7.7), we have

$$\boldsymbol{N} \cdot [\boldsymbol{Q}] = \boldsymbol{N} \cdot [E\bar{\boldsymbol{V}} + \boldsymbol{T} \cdot \boldsymbol{v} + \mathfrak{M} \cdot \dot{\mathfrak{X}}] = \boldsymbol{N} \cdot [\theta S \bar{\boldsymbol{V}}] - q_\Sigma, \tag{7.14}$$

where the latter follows from (7.9). Thus,

$$q_\Sigma = -\boldsymbol{N} \cdot [W\bar{\boldsymbol{V}} + \boldsymbol{T} \cdot \boldsymbol{v} + \mathfrak{M} \cdot \dot{\mathfrak{X}}], \tag{7.15}$$

where we performed the Legendre transformation $W = E - S\theta$. But because of the continuity conditions (7.1 *b*) and (7.2), and of the second relation of (2.6), we immediately check that

$$\boldsymbol{N} \cdot [\boldsymbol{T} \cdot \boldsymbol{v} + \mathfrak{M} \cdot \dot{\mathfrak{X}}] = -\boldsymbol{N} \cdot [\boldsymbol{T} \cdot \boldsymbol{F} + \mathfrak{M} : \nabla_{\mathrm{R}} \mathfrak{X}] \cdot \bar{\boldsymbol{V}}. \tag{7.16}$$

On comparing (7.15) and (7.13), we find that

$$P(\boldsymbol{f}_\Sigma) = \boldsymbol{f}_\Sigma \cdot \bar{\boldsymbol{V}} = q_\Sigma. \tag{7.17}$$

As we are in quasi-statics and the jump relations (7.5) and (7.6) reduce to

$$\boldsymbol{N} \cdot [\boldsymbol{T}] = \boldsymbol{0}, \quad \boldsymbol{N} \cdot [\boldsymbol{M}] = \boldsymbol{0}, \tag{7.18}$$

we note that

$$\boldsymbol{N} \cdot [\boldsymbol{T} \cdot \boldsymbol{F} + \mathfrak{M} : \nabla_{\mathrm{R}} \mathfrak{X}] = \langle \boldsymbol{N} \cdot \boldsymbol{T} \rangle \cdot [\boldsymbol{F}] + \langle \boldsymbol{N} \cdot \mathfrak{M} \rangle : [\nabla_{\mathrm{R}} \mathfrak{X}], \tag{7.19}$$

since $\mathfrak{X}$ is obviously continuous across Σ. Here the symbolism $\langle \cdot \rangle$ denotes the average of the enclosed quantity at Σ. Finally, using the Maxwell–Hadamard lemma for jumps of gradients of $\boldsymbol{F}$ and $\mathfrak{X}$, according to which $[\boldsymbol{F}]$ and $[\nabla_{\mathrm{R}} \mathfrak{X}]$ have representations

$$[\boldsymbol{F}]^i_{\cdot K} = f^i N_K, \quad [\nabla_{\mathrm{R}} \mathfrak{X}]^i_{\cdot KL} = g^i_{\cdot K} N_L, \tag{7.20}$$

we get

$$\langle \boldsymbol{N} \cdot \boldsymbol{T} \rangle \cdot [\boldsymbol{F}] \cdot \bar{\boldsymbol{V}} + \langle \boldsymbol{N} \cdot \mathfrak{M} \rangle : [\nabla_{\mathrm{R}} \mathfrak{X}] \cdot \bar{\boldsymbol{V}} = [\mathrm{tr}\{\langle \boldsymbol{T} \rangle \cdot \boldsymbol{F} + \langle \mathfrak{M} \rangle : \nabla_{\mathrm{R}} \mathfrak{X}\}] \bar{V}_N. \tag{7.21}$$

Gathering these results we obtain that

$$q_\Sigma = -\mathcal{H}_G \bar{V}_N \geqslant 0, \tag{7.22}$$

where we have defined the so-called Gibbs–Hugoniot driving force $\mathcal{H}_G$ by

$$\mathcal{H}_G = \boldsymbol{N} \cdot [\boldsymbol{b}] \cdot \boldsymbol{N} = [W - \mathrm{tr}\{\langle \boldsymbol{T} \rangle \cdot \boldsymbol{F} + \langle \mathfrak{M} \rangle : \nabla_{\mathrm{R}} \mathfrak{X}\}]. \tag{7.23}$$

The results of this section show the following.

(i) If there is progress of the transition front, $\bar{V}_N \neq 0$, the dissipation at the front is manifested by a localized *hot* surface source q_Σ.

(ii) The Eshelby stress plays a fundamental role in computing the driving traction acting on the front. Denoting by f_Σ the normal component of $\boldsymbol{f}_\Sigma$, we can write the following balance of material forces at Σ:

$$f_\Sigma + \mathcal{H}_G = 0.$$

This equation represents the competition between a field quantity $\mathcal{H}_G$—that is known at each instant when the field solution is known on both sides of Σ; this by any means, analytical or numerical—and the constitutive quantity f_Σ, which is constrained to satisfy the dissipation condition

$$\bar{\theta} d_\Sigma = f_\Sigma \bar{V}_N \geqslant 0. \tag{7.24}$$

The latter provides a criterion of progress of the front, in a way similar to what happens in elasto-plasticity, once we have a relationship

$$\bar{V}_N = \mathfrak{B}(f_\Sigma, \bar{\theta}), \tag{7.25}$$

which is of the threshold type (i.e. $\bar{V}_N = 0$ if $|f_\Sigma|$ has not reached the critical value or $\bar{V}_N$ possibly non-zero when this critical value is reached). More sophisticated criteria that satisfy the constraint (7.24) can be imagined. In good physics, however, such criteria should follow from a local analysis of the transition phenomenon at a smaller scale.

(iii) The peculiar form (the second equation of (7.23)) of $\mathcal{H}_G$—that does not involve $\boldsymbol{N}$—follows from the working hypothesis of quasi-statics. However, the fact that $\mathcal{H}_G$, in agreement with Gibbs & Duhem's vision, does not involve kinetic energy, for it governs local material structural changes only, is a general fact holding even when inertia is kept in the development (forthcoming). The proof then simply is a little more lengthy. On account of inertia, the result would be

$$\mathcal{H}_G = [W] - \langle \boldsymbol{N} \cdot \boldsymbol{T} \rangle \cdot \boldsymbol{f} - \langle \boldsymbol{N} \cdot \mathfrak{M} \rangle \cdot \boldsymbol{g}, \tag{7.26}$$

where $\boldsymbol{f}$ and $\boldsymbol{g}$ are the two two-point fields introduced component wise in equations (7.20).

(iv) The condition $\mathcal{H}_G = 0$ (no dissipation) obviously resembles the Hugoniot condition (also an equation obtained through manipulations and not a field equation *per se*) of shock-wave theory which selects the manner in which the two states of a material on both sides of the wave are connected. However, whereas it is internal energy that is involved in shock studies, it is the free energy that appears in our $\mathcal{H}G$. The reason for that is to be found in the basic hypothesis that here temperature is continuous across Σ, while entropy is the determining thermodynamical quantity in discussing shock waves.

8. Small strains and rotations

This special case is of interest because it is the only framework in which expressions have been given so far for material forces in polar elastic solids; in fact, in a disguise, so that it affords comparisons. In this case, one naturally introduces the relative Cosserat tensor by

$$\mathfrak{E} := \mathfrak{C} - \mathbf{1}_{\mathrm{R}}. \tag{8.1}$$

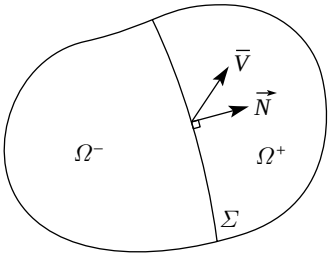


Figure 2. Phase-transition front Σ.

The motion gradient $\boldsymbol{F}$ and micropolar rotation $\mathfrak{X}$ can be approximated by

$$\boldsymbol{F} = \mathbf{1}_{\mathrm{S}} + (\nabla_{\mathrm{R}}\boldsymbol{u})^{\mathrm{T}}, \quad \mathfrak{X} = \mathbf{1}_{\mathrm{S}} + \boldsymbol{\Phi}, \tag{8.2}$$

where $\mathbf{1}_{\mathrm{S}}$ is a so-called shifter, $\boldsymbol{u}$ is the displacement field and $\boldsymbol{\Phi}$ is skewsymmetric . Accordingly, we have the following approximations:

$$\mathfrak{C} \cong \mathbf{1}_{\mathrm{R}} + (\nabla_{\mathrm{R}}\boldsymbol{u})^{\mathrm{T}} + \boldsymbol{\Phi}, \quad \mathfrak{E} \cong (\nabla_{\mathrm{R}}\boldsymbol{u})^{\mathrm{T}} + \boldsymbol{\Phi}. \tag{8.3}$$

Simultaneously, for $\mathfrak{G}$ we have the linearized measure

$$\mathfrak{G} \cong \tfrac{1}{2}\mathbf{1}_{\mathrm{S}}\dot{\times}(\nabla_{\mathrm{R}}\boldsymbol{\Phi}). \tag{8.4}$$

Define ϕ, the vector dual to the skewsymmetric tensor $\boldsymbol{\Phi}$, by

$$\phi_K = \tfrac{1}{2}\varepsilon_{KLM}\boldsymbol{\Phi}_{ML}, \quad \boldsymbol{\Phi}_{KL} = -\varepsilon_{KLM}\phi_M. \tag{8.5}$$

Then, instead of $\mathfrak{E}$ and $\mathfrak{G}$, we can, as well, use the following infinitesimal measures of deformation for a linear polar elastic solid (we no longer distinguish between lower and upper case Latin indices):

$$\mathfrak{e} := (\nabla\boldsymbol{u})^{\mathrm{T}} + \mathrm{dual}\,\phi = \{\mathfrak{e}_{ij} = u_{j,i} - \varepsilon_{ijk}\phi_k\}, \tag{8.6}$$

$$\gamma := \nabla\phi = \{\gamma_{ij} = \phi_{i,j}\}. \tag{8.7}$$

In this approximation, the tensors $\boldsymbol{T}$ and $\boldsymbol{M}$ reduce to the usual (but here non-symmetric) Cauchy stress tensor $\boldsymbol{t}$ and couple-stress tensor $\boldsymbol{m}$ with constitutive equations

$$\boldsymbol{t} = \frac{\partial\hat{W}}{\partial\mathfrak{e}}, \quad \boldsymbol{m} = \frac{\partial\hat{W}}{\partial\gamma}, \quad S = -\frac{\partial\hat{W}}{\partial\theta}, \tag{8.8}$$

where, for a materially inhomogeneous thermoelastic polar solid, we have a free energy density given by

$$W = \hat{W}(\mathfrak{e}, \gamma, \theta; \boldsymbol{x}). \tag{8.9}$$

The two basic laws of motion (5.11) and (5.12) take on the following form (compare to Eringen 1968; Nowacki 1986):

$$\rho_0\frac{\partial^2\boldsymbol{u}}{\partial t^2} - \div\boldsymbol{t} = \mathbf{0}, \quad \rho_0 j\cdot\frac{\partial^2\phi}{\partial t^2} - \div\boldsymbol{m} - (\mathbf{1}\times\boldsymbol{t}) = \mathbf{0}. \tag{8.10}$$

In components, the last equation reads

$$\rho_0 j_{ij}\left(\frac{\partial^2\phi_j}{\partial t^2}\right) - \left(\frac{\partial m_{ji}}{\partial x_j}\right) - \varepsilon_{ipq}t_{pq} = 0. \tag{8.11}$$

Isotropic microinertia $j = I\mathbf{1}$, i.e. $j_{ij} = I\delta_{ij}$, is often assumed but this restriction is not imposed by the formulation.

On account of the simplifications introduced above, the canonical balance equations (energy and pseudo-momentum) take on the following form:

$$\frac{\partial}{\partial t}(\rho_0(e+\tfrac{1}{2}\dot{\boldsymbol{u}}^2+\tfrac{1}{2}\dot{\phi}\cdot j\cdot\dot{\phi}))-\nabla\cdot(\boldsymbol{t}\cdot\dot{\boldsymbol{u}}+\boldsymbol{m}\cdot\dot{\phi}-\boldsymbol{q})=0 \tag{8.12}$$

and

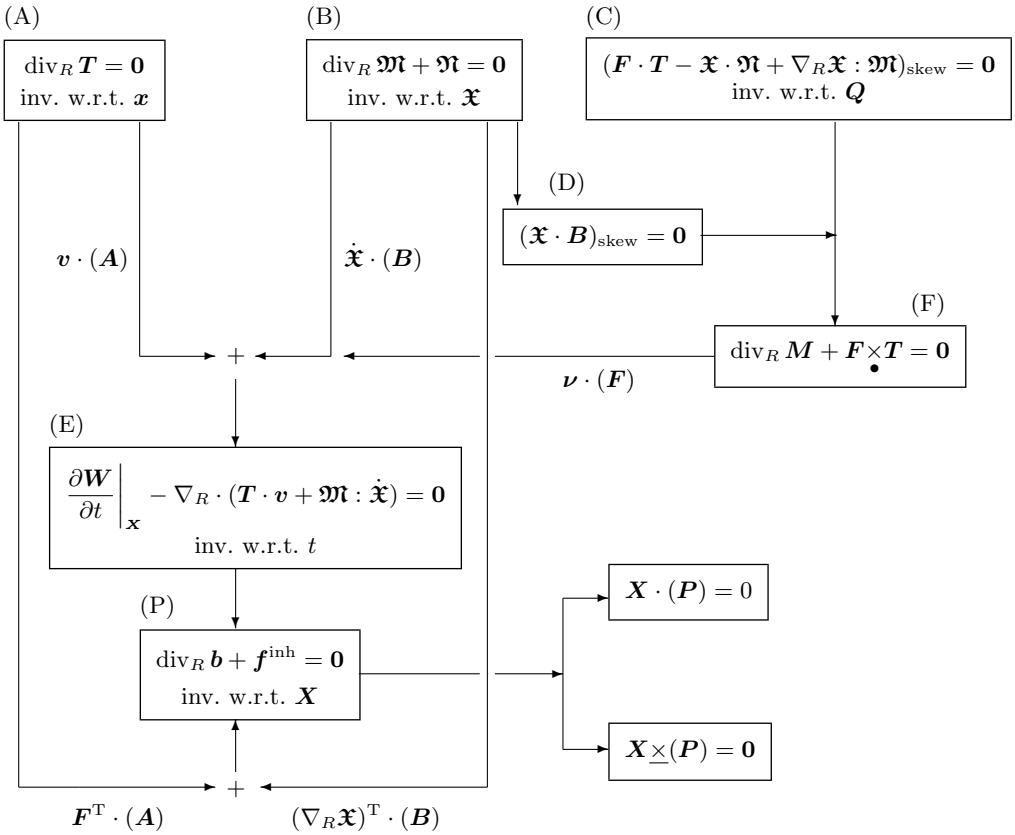
$$\frac{\partial}{\partial t}(-\rho_0(\nabla\boldsymbol{u})\cdot\dot{\boldsymbol{u}}-\rho_0(\nabla\phi)\cdot j\cdot\dot{\phi})+\left(\frac{\partial\hat{W}}{\partial\boldsymbol{x}}\right)_{\text{expl}}-S\nabla\theta$$
$$-\div((\hat{W}-\tfrac{1}{2}\rho_0\dot{\boldsymbol{u}}^2-\tfrac{1}{2}\rho_0\dot{\phi}\cdot j\cdot\dot{\phi})\mathbf{1}-\boldsymbol{t}\cdot(\nabla\boldsymbol{u})^{\mathrm{T}}-\boldsymbol{m}\cdot(\nabla\phi)^{\mathrm{T}})=\mathbf{0}. \tag{8.13}$$

Now the application to fracture (cf. § 4) is straightforward. In quasi-statics, the projection of $\mathcal{F}(A)$ onto the direction $\boldsymbol{E}_1$ of possible extension of the crack yields the J integral of the brittle-fracture mechanics of (materially homogeneous) polar elastic media as

$$J(A)=\mathcal{F}(A)\cdot\boldsymbol{E}_1=\int_{\Gamma}\left(Wn_1-\boldsymbol{n}\cdot\left(\boldsymbol{t}\cdot\frac{\partial\boldsymbol{u}}{\partial X}+\boldsymbol{m}\cdot\frac{\partial\phi}{\partial X}\right)\right)\mathrm{d}\Gamma, \tag{8.14}$$

with $W=W(\mathfrak{e},\gamma)$, $\partial/\partial X=\boldsymbol{E}_1\cdot\nabla$, $n_1=\boldsymbol{n}\cdot\boldsymbol{E}_1$, where $\boldsymbol{n}$ is the unit outward normal to any circuit Γ that surrounds the tip A of the crack whose faces are totally free of loads. The integral J in equation (8.14) is contour independent, just like its classical elasticity analog due, among others, to Rice. This is due to the canonical expression of the balance of pseudo-momentum. We may consider that an expression such as (8.14) is essentially contained in the conservation law (5.6)—which does not concern cracks—of Jaric (1978). However, equation (8.12) contains also the germ of the generalization of this result to (i) dynamics, (ii) the thermoelasticity of conductors and (iii) materially inhomogeneous polar elastic media. To do this we simply need to follow the prescriptions given in previous works (e.g., respectively, in Dascalu & Maugin 1993; Dascalu & Maugin 1995; Maugin 1995, pp. 228–229 for these three cases). For example, for the dynamics of purely elastic polar solids, on account of (8.13) simplified in the absence of true-inhomogeneity and thermal effects, we note that the space integral of (8.13) over a regular material domain yields the conservation law (5.8) of Vukobrat (1989), obtained by this author directly in the linear framework, although we have here at our disposal the general proof in the nonlinear framework and with both material inhomogeneities and thermal effects included. The corresponding J integral of fracture will, therefore, include not only a surface term (in two dimensions) which is transformed into a contour integral, but also a bulk term (a surface integral for two-dimensional problems) involving the pseudo-momentum. We shall not repeat for polar bodies the argument of Dascalu & Maugin (1993) for it would directly lead from the integral of (8.13) over a material volume about the crack to the J integral (6.1) of Vukobrat (1989, p. 1102) (note that in the two-dimensional problem the only relevant component, ϕ_3, of ϕ is in a direction orthogonal to the plane) which is still contour-surface independent.

It is not salient to give the expressions of linear elasticity for phase-transition fronts as it appears that the finite-strain framework is practically always required and nobody seems to have given corresponding expressions by another means before.



$$W = W(\boldsymbol{F}, \mathfrak{X}, \nabla_r\mathfrak{X}; \boldsymbol{X}), \quad \boldsymbol{f}^{\text{inh}} = -\left(\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{X}}\right)_{\text{expl}}$$

$$\boldsymbol{T} = \left(\frac{\partial \boldsymbol{W}}{\partial \boldsymbol{F}}\right)^{\mathrm{T}}, \quad \mathfrak{M} = \left(\frac{\partial \boldsymbol{W}}{\partial \nabla_R \boldsymbol{X}}\right)^{\mathrm{T}}, \quad \mathfrak{N} = -\left(\frac{\partial \boldsymbol{W}}{\partial \mathfrak{X}}\right)^{\mathrm{T}}$$

$$\boldsymbol{b} = \boldsymbol{W}\boldsymbol{1}_R - \boldsymbol{T}\cdot\boldsymbol{F} - \mathfrak{M} : \nabla_R\mathfrak{X}, \quad \mathfrak{M} : \dot{\mathfrak{X}} \equiv \boldsymbol{M}\cdot\boldsymbol{\nu}$$

Figure 3. Flow chart of conservation laws for polar media.

9. Conclusion

The main aim of this lengthy contribution was to delineate the general structure of the theory of polar elastic materials, especially in so far as canonical balance laws and their applications are concerned. This is summarized in figure 3 which shows a flow chart of the connections between these local balance laws. Essential applications are those to the theories of fracture and of the propagation of phase-transition fronts, for which progress criteria constitute the relevant practical objectives. In this line of thought there was no attempt at showing the completeness of the set of balance laws obtained. We preferred to put the emphasis on those laws which certainly play the most important role in engineering applications despite their deep physical significance. It is believed that a state of generality has been reached that has

certainly not been attained so far, including, in the finite-deformation framework, the theory of material inhomogeneities and the effects of heat conduction . The latter are indeed essential in the correct formulation of the driving force that acts on progressing coherent phase-transition fronts. This state of generality can only be grasped because of the canonical structure of the chosen type of approach. In some sense, the general results can be read off field theory. This is what simultaneously provides a guarantee of the validity of the obtained results and an aesthetic pleasure to which it is hard to resist. In pursuing this line, which does not necessarily bring immediate rewards, it should be emphasized that in view of the recurrent interest for a geometrical theory of defects—these will be both dislocations and disclinations in the present context as the basic mechanical model certainly allows for them—the next effort should be devoted to the possible geometrization of the canonical balance laws of pseudo-momentum and moment of pseudo-momentum, equations (3.6) and (3.12) in the quasi-static approximation. We remind the reader that this ambitious programme—which is not without recalling Einstein's successful ambition to incorporate some physical field source in the geometry of the underlying space—normally consists in devising a geometry on the material manifold such that the material balance laws take a more compact form. Typically, one would like to incorporate the effects of quasi-inhomogeneities (i.e. fields such as those related to density of defects in interaction with the deformation field) in a certain covariant divergence of the stress field (here the Eshelby stress) and possibly of the field of material hyperstresses (cf. equation (3.12)). This problem was solved by Epstein & Maugin (1990) in the pure classical elasticity of continuously dislocated bodies. Promising attempts at generalizing this for the elasticity of the second gradient were given by Elzanowski *et al.* (1990) and de Leon & Epstein (1995). These rely on the general theory of material uniformity put forth by these authors. The basic properties related to the differential geometry of Cosserat continua, the object of the present contribution, were examined by Epstein & de Leon (1994), while anelasticity is viewed as evolving quasi-inhomogeneities by Epstein & Maugin (1995). The endeavour presented by this 'geometrization' of equations (3.6) and (3.12) has to be completed in the near future. Its difficulty, however, is of the same level as that of the programme set forth by Kröner & Maugin (1998) in anelastic solids in large deformation. It appears, thus, that microstructure of the type met in (non-dissipative) polar elastic solids and inelasticity of an otherwise classical elastic continuum yield difficulties and solutions of the same order. In the precise case of polar media, the invariance properties originally established by Kafadar (Eringen & Kafadar 1976, pp. 18–20) will be essential in such developments.

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